

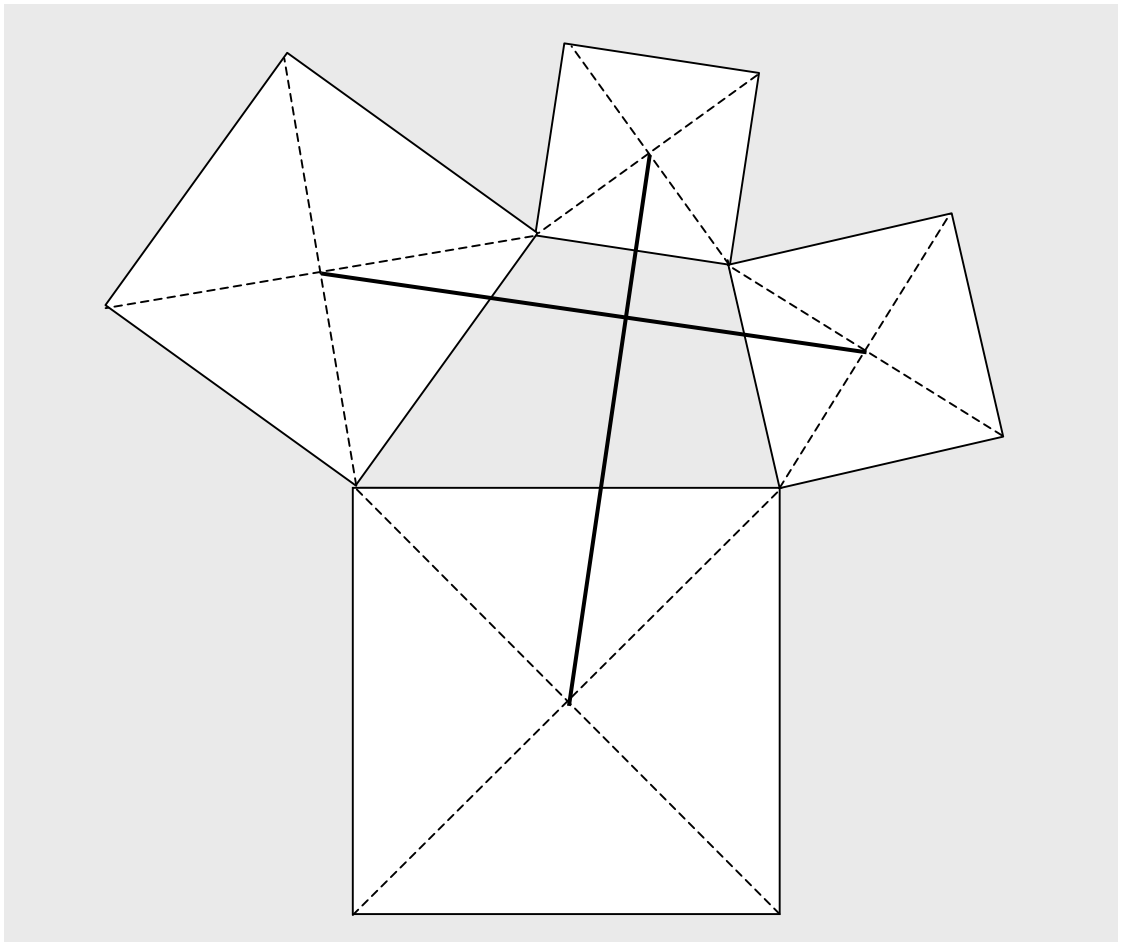
# Van Aubel's Theorem

Copyright © 2018 by Seong Ryeol Kim. All rights reserved

Click this to start.

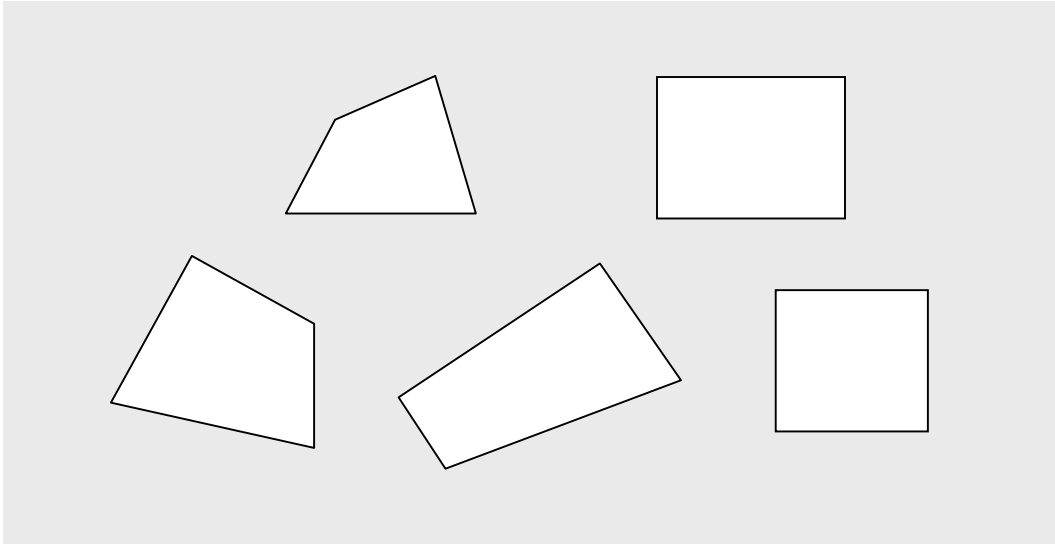
# Van Aubel's Theorem Part 1

In the construction of this lesson, a reference was made to a journal published by Yutaka Nishiyama, a math professor at Osaka University of Economics, Osaka, Japan.

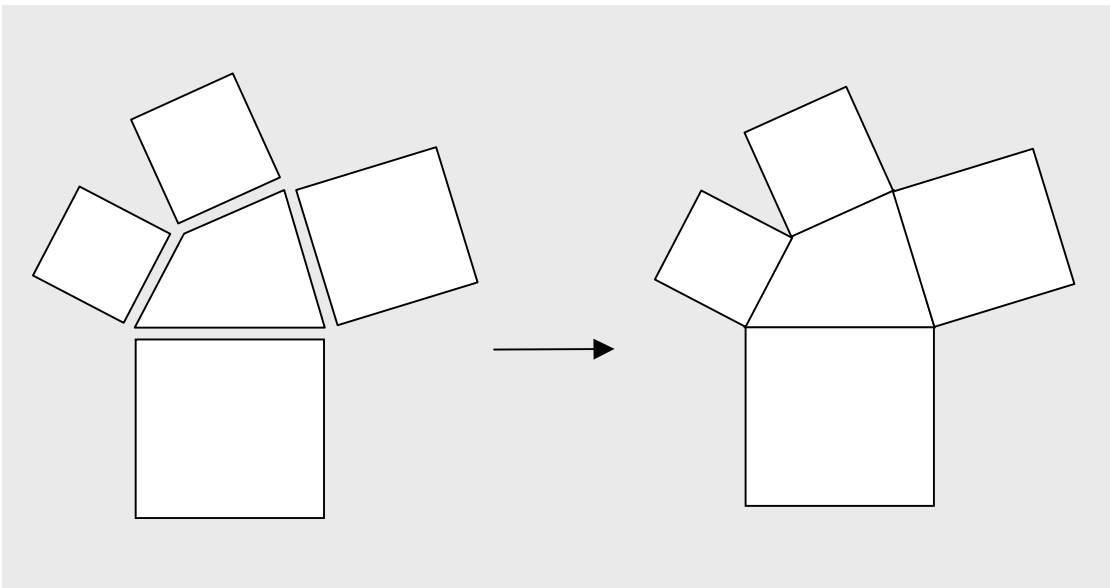


The journal is about Van Aubel's Theorem (1878), and explains the proof using elementary geometry. And the theorem goes the way as follows.

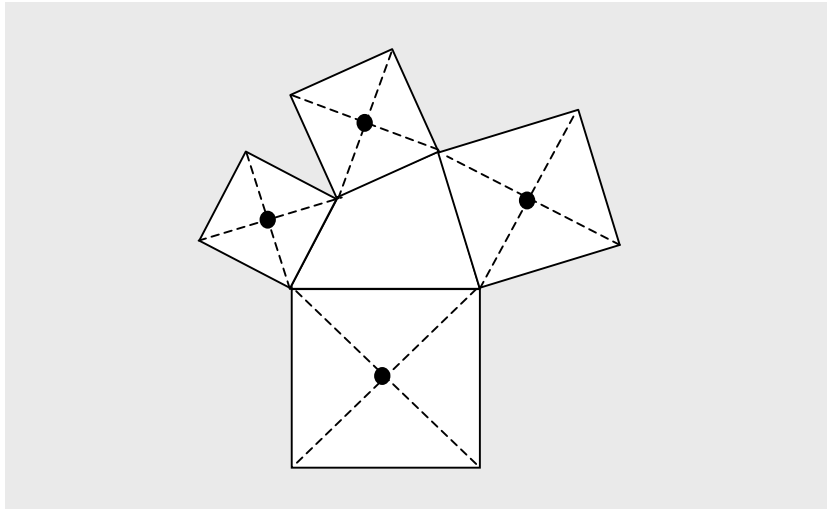
First, pick a quadrangle from the ones shown below.



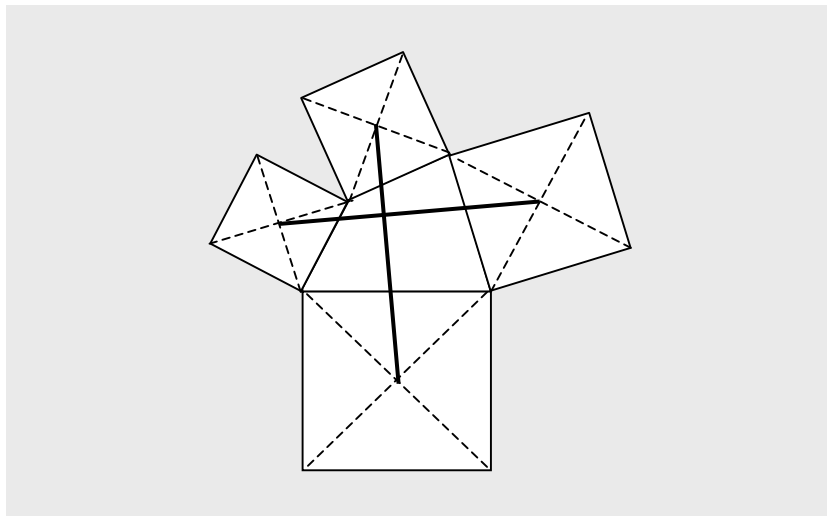
Of the five above, let's pick the one on the right at the top, since it looks quite general and easy to work with. Then, construct squares around it, and put them on its sides the way below.



Next, put a dot at the center of each square.

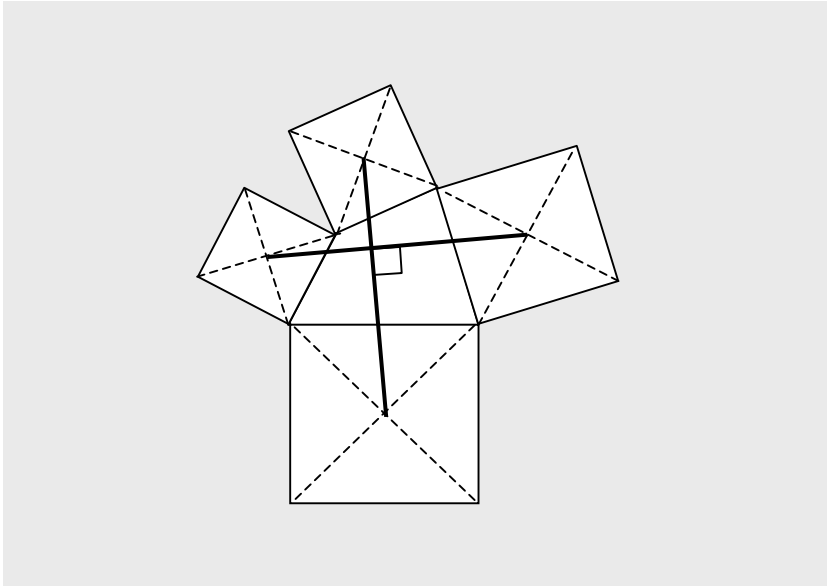


And now, connect the two dots facing each other across the quadrangle using a line segment. Then, we get this:



So we get two line segments crossing each other. And the two have some lengths, and meet at an angle.

Now, the question is, what lengths and what angle?



The theorem says that the two line segments meet at a right angle, that is,  $90^\circ$ , and have the same lengths.

They are of the same length and perpendicular to each other. And the same is true of any other quadrangle, too.

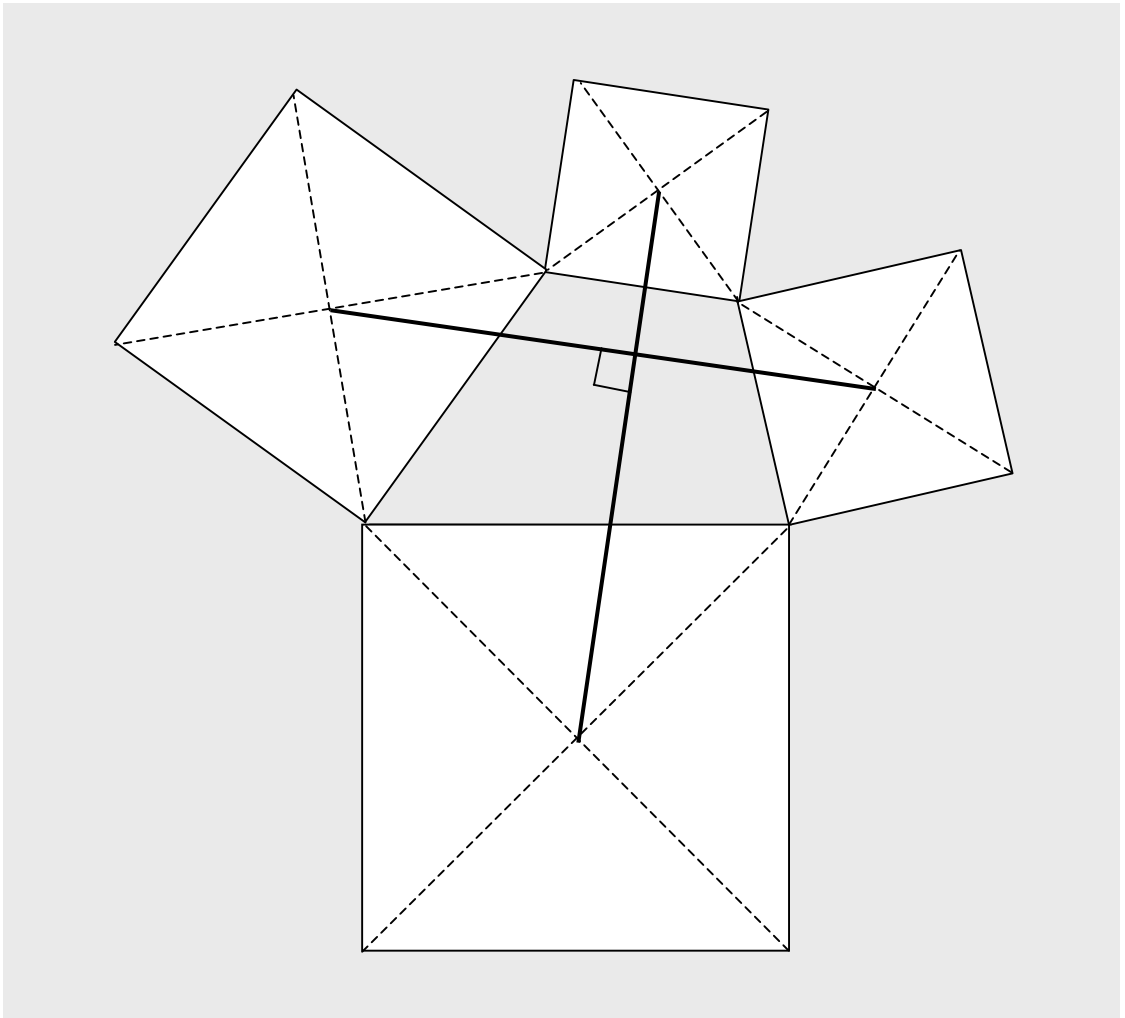
So whatever the quadrangle may be, the two line segments are ***always*** the same as and perpendicular to each other.

“How?”, I wondered.

Some math was done, with some basics running, and the reason showed up.

You might wonder why, too. So after these lessons, you get to see also, why always the same and  $90^\circ$  in the story above. And the story is Van Aubel's Theorem.

Again, the theorem says, as shown below, the two line segments share a length and make  $90^\circ$  no matter what the quadrangle may be.



In mathematics, we call ***how it is the case*** the proof.

And you gonna see the proof, or rather, you gonna get it. 😊

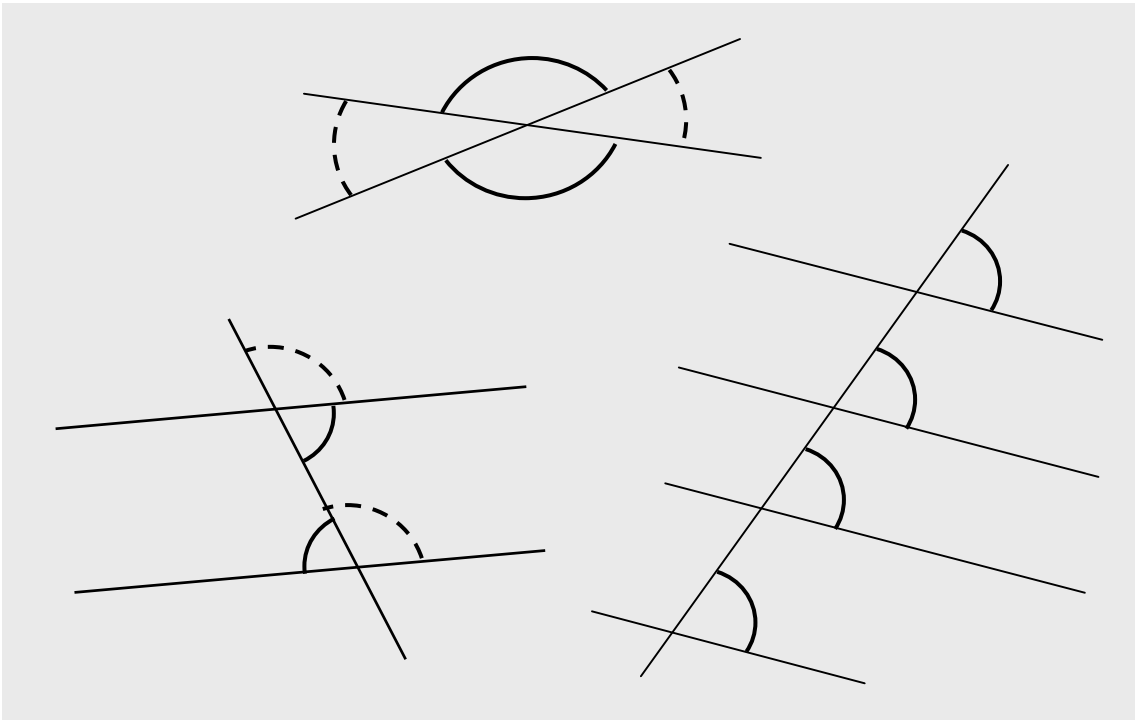
So it'll be all yours, after all.

Of course, you not gonna be asked to produce it yourself.

You might be able to come up with it all by yourself, though.

This lesson will help you get the proof. The proof is however, not the point. What then, is the point?

The point is how the proof is made, or rather, what's behind the proof. What then is behind it?

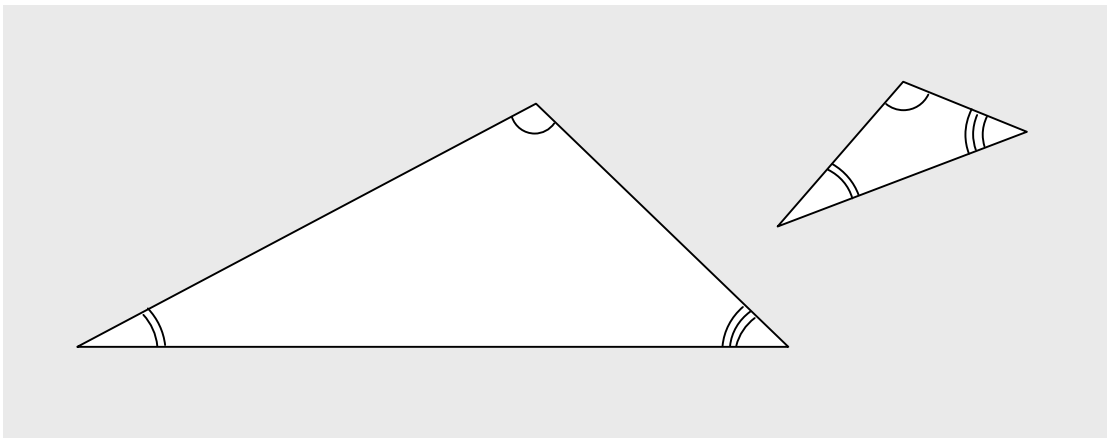


It is the basics, the math basics used in the proof.

And that's what matters to most of you not mathematicians.

So what matters is the basics used in the construction of the proof, and what does really do matter is that you know how to use the basics, so that you can use them doing problems.

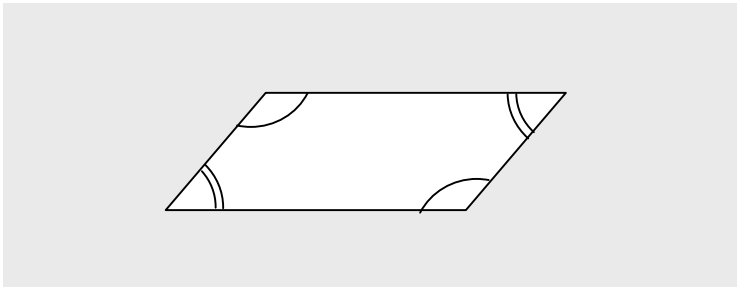
What basics, though?



They are math basics often used when we find solutions to many problems. So you want to know them very well.

And the basics covered in these lessons are as follows.

- Corresponding angles with parallel lines are equal.
- Alternate angles with parallel lines are equal.
- Vertical angles are equal.
- The sum of all the three angles in a triangle
- The sum of all the four angles in a tetragon
- The conditions for similar triangles
- The conditions for identical triangles
- The properties of a parallelogram, and some others



Well, they are in fact, sort of prerequisites.

The lessons have been designed with the assumption that you have some knowledge of the basics listed above.

Relax, though, even if you don't have much of it or any at all. You can still get much out of all the lessons, since they can be good places where you start knowing the basics useful.

Knowing them very well, you don't really need these lessons.  
Why not?

Knowing them very well, you know what they are about, how  
they work, how to use them, and what to do with them.

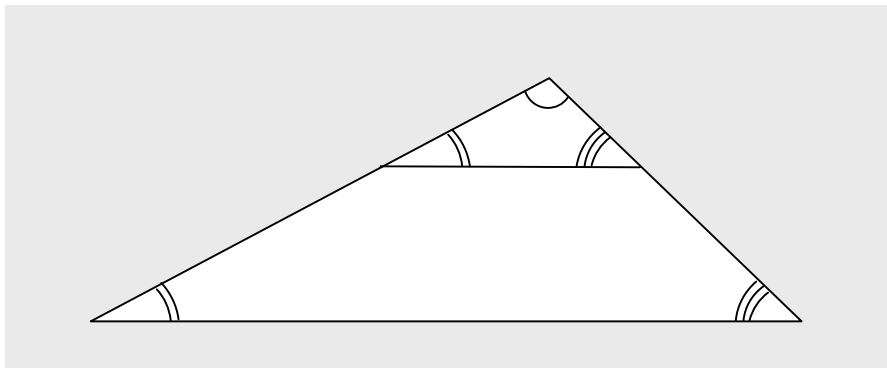
Knowing them very well, therefore, you'd better move on.

These lessons have been designed to help understand **how  
those math basics can be applied** in problem solving  
processes, so that you can use them effectively when doing  
many problems, as well as doing the proof of this theorem.

Why basics, though? Basics are easy, aren't they?

So are they worth the effort? Why bother?

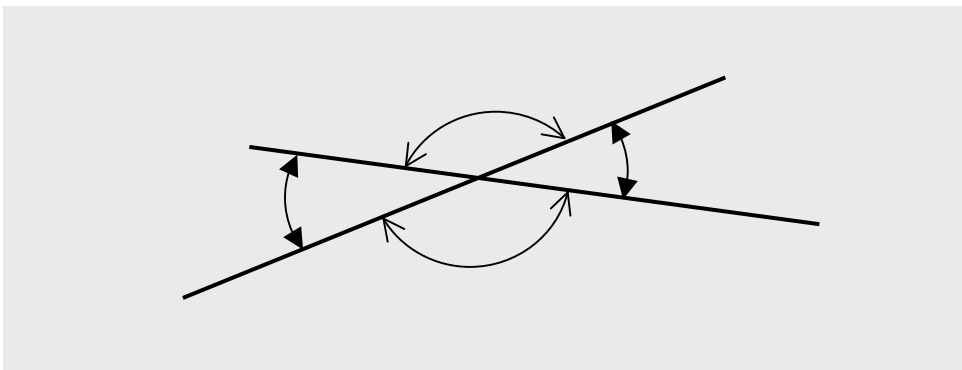
Why not technical or tricky tricks instead, so that you can do  
problems fast?



Of course, basics should not be hard. And they should be easy to use. It doesn't seem, though, they are quite handy and easy for many students. What about you?

Are they handy and easy for you when you solve problems?

If so, you don't need these, because you've already owned these: how the basics work, how to use them, and what to do with them. You've already done with all these lessons.



So here in these lessons, you get to own those listed above actually and practically, so that they can be convenient tools you can easily and effectively work with when doing problems and learning things in math.

So, together with other basics, they can be working tools you love to use when running your math. Running math?

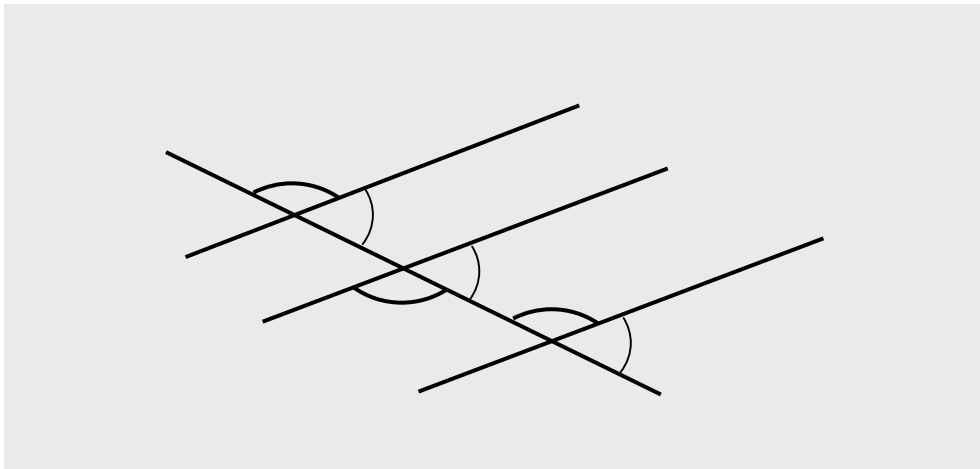
Math is made of a lot of operations, along with many ideas. And running math, you solve problems. So solving problems, you don't just do math, you run your own math.

So you want to run your math the way competent people run their own businesses. They want their businesses to run properly. So do you.

Your math business can run properly if your tools work. If you handle them effectively, they do the jobs fast. Math is a reasoning business where you reason to solve problems.

This lesson is designed to help you handle the basics effectively and with ease.

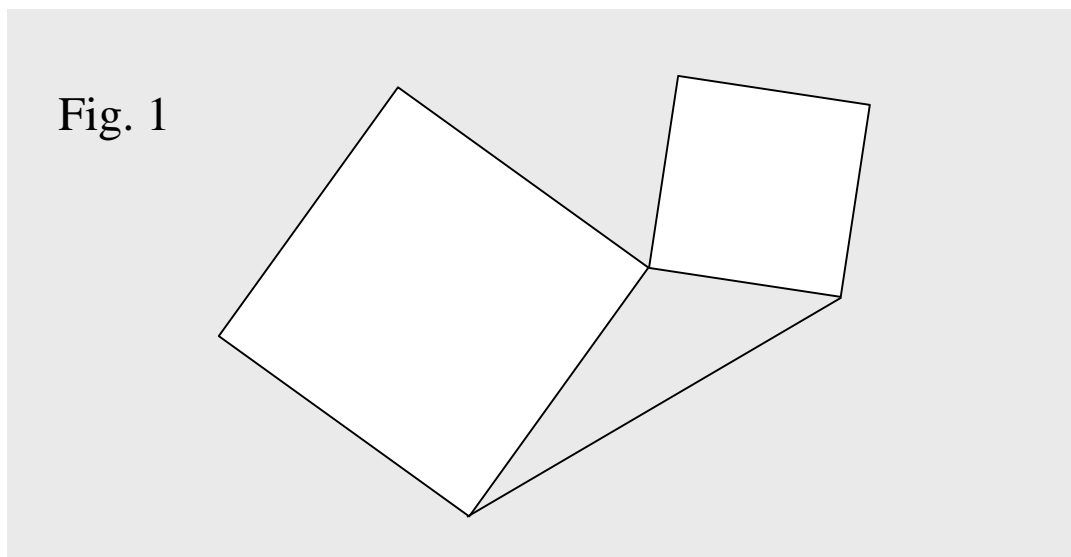
In short, you get to make the basics wieldy.



The statement, *always the same* and  $90^\circ$  stated in the theorem isn't easy or straightforward for many students to prove, and it's quite involved even for advanced students. It can be easier, though. How?

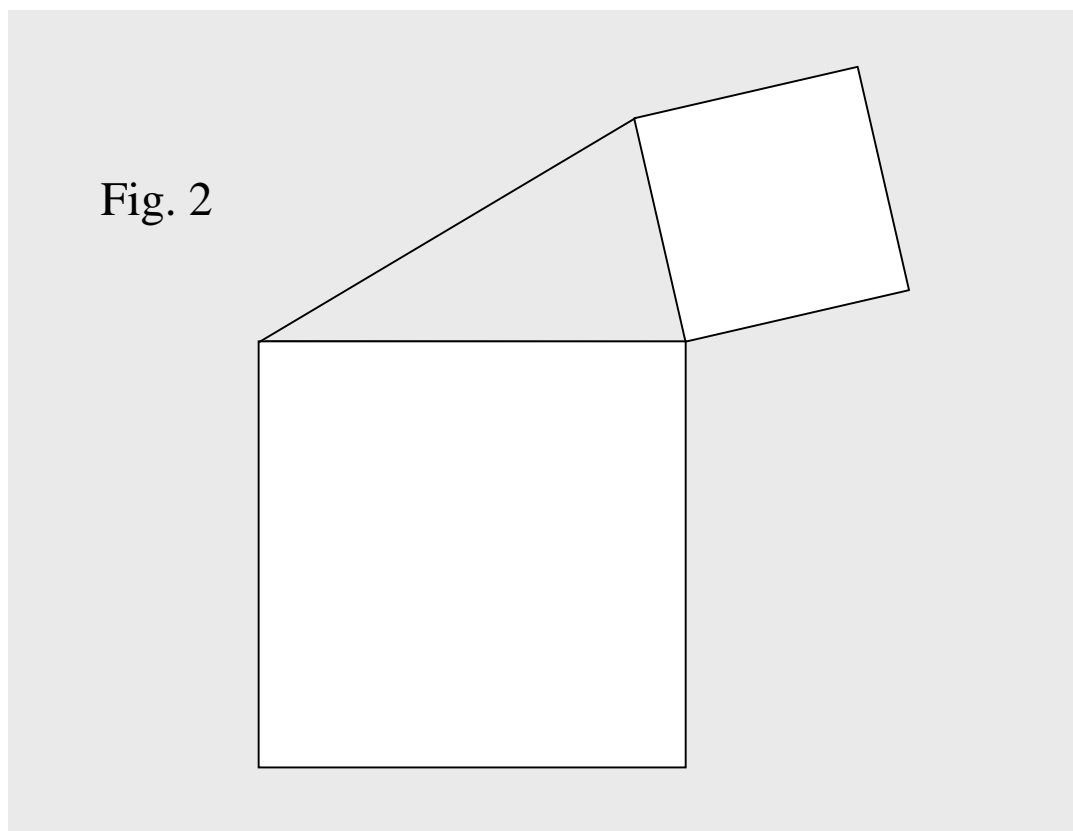
We can take it as a problem, because the proof can be taken as a solution. We can break it into parts so that the parts can be more doable, that is, easier to solve; thus, the problem gets easier. So now, this lesson begins with a part of the problem as follows.

Putting together two squares and a triangle, we can make a figure shown below.



Why that figure, though? What's the idea?

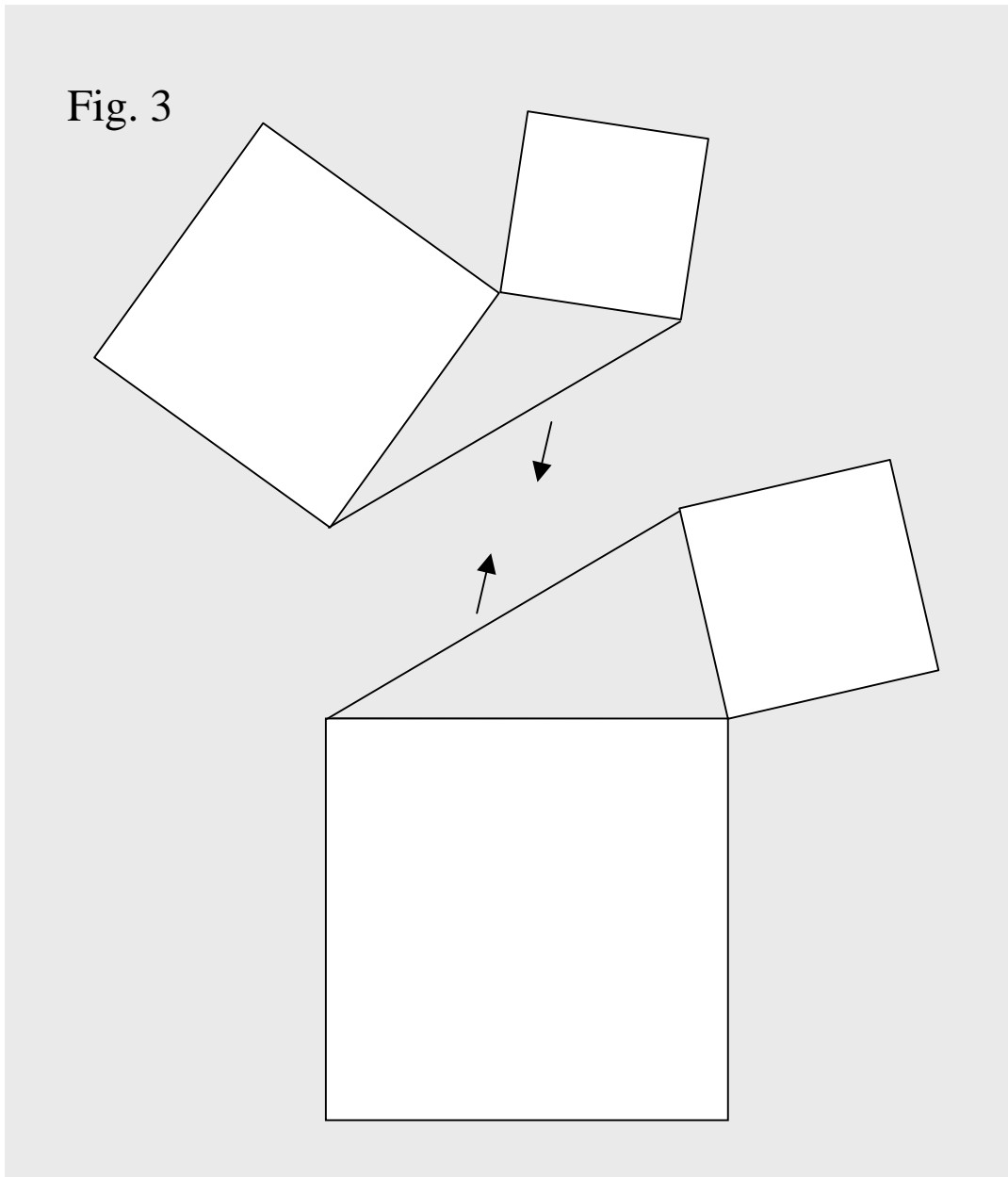
The figure above is no other than the figure below.



So putting together again, two squares and a triangle, we can make a figure shown above, too.

And putting together those two figures constructed above, we can get the whole figure presented in the theorem.

So now, putting the two figures together the way as follows, we will get the whole figure in the theorem.



We can call the method above divide-and-concur.

Putting the two figures together, we can get this:

Fig. 4

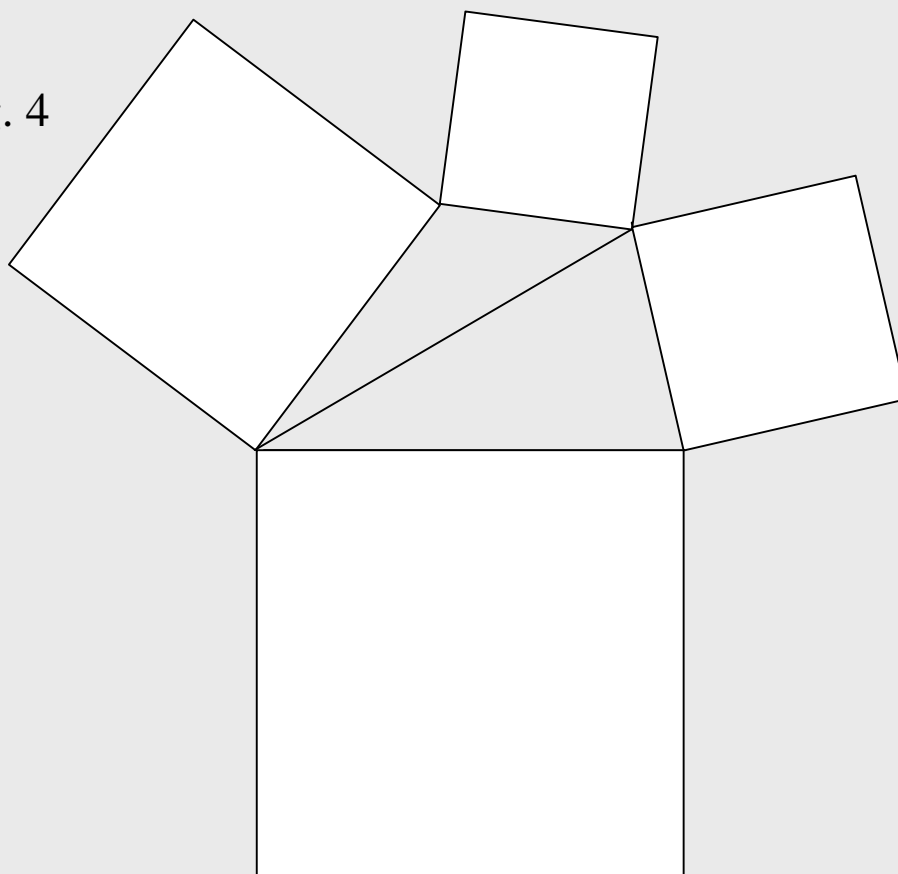
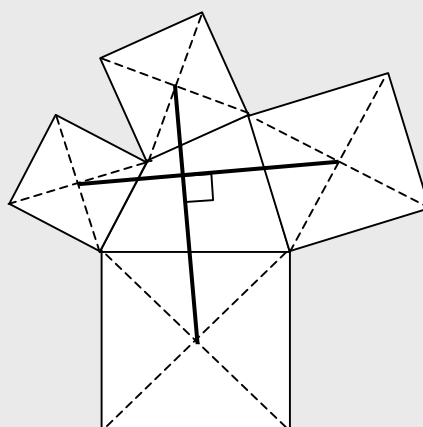
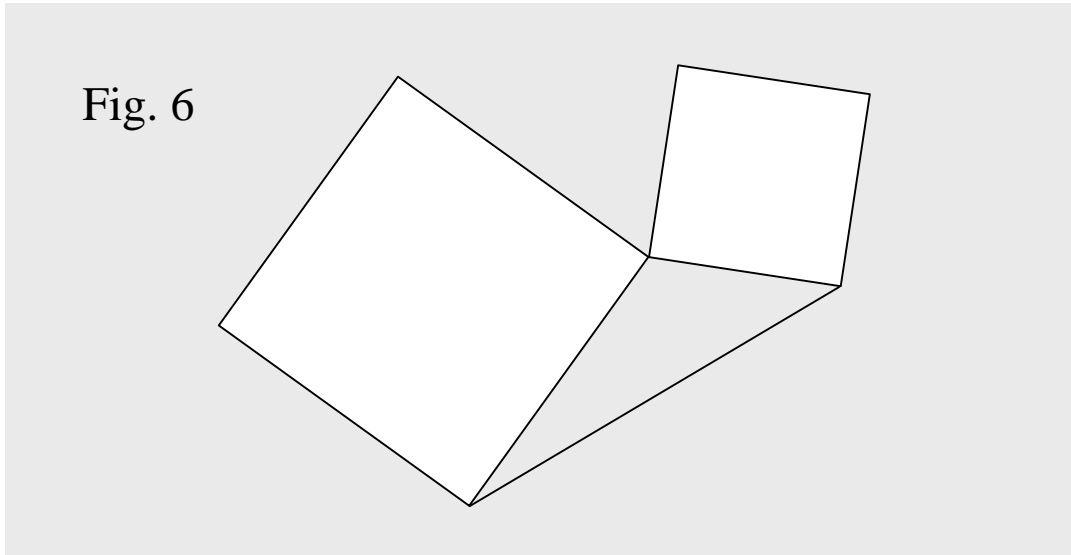


Fig. 5



So solving the figure below, we can easily get the solution to the other figure. And putting the two solutions together, we can approach more easily the solution to the whole problem.

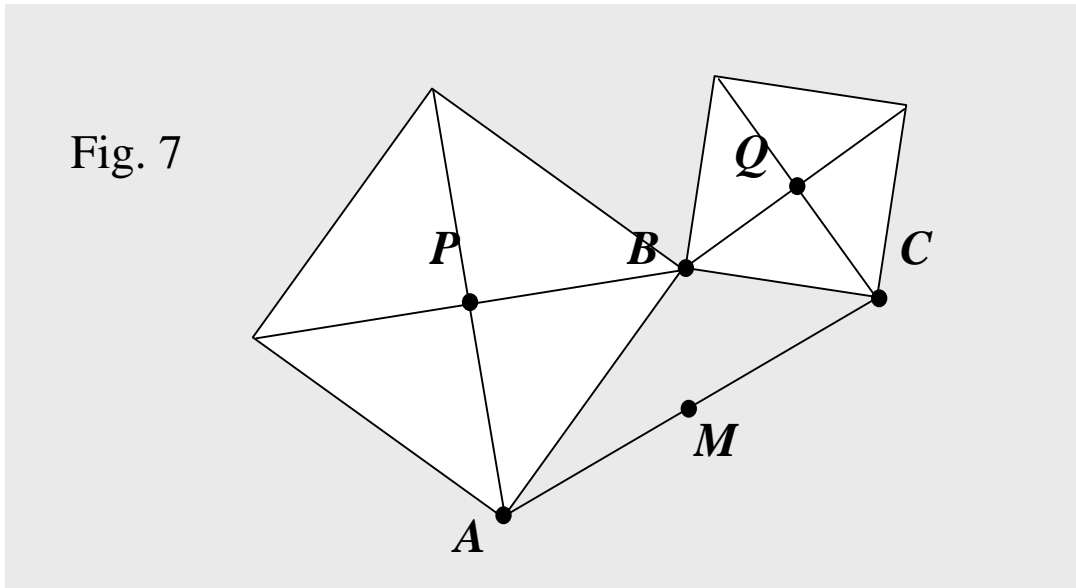


Let's now, start getting the solution to the one above.

The theorem states that a line segment connects the centers of the squares. So let's now put in the figure above, two points at the centers of the two squares, and name the triangle naming the vertices. Naming objects, we can easily refer to them. So name them first. Name them all.

And naming them in math, we often use letters, alphabets.

So now, in the figure below,  $P$  and  $Q$  are the centers of the two squares,  $A$ ,  $B$ , and  $C$  are the vertices of the triangle, and  $M$  is the midpoint of the side  $AC$  in  $\triangle ABC$ .



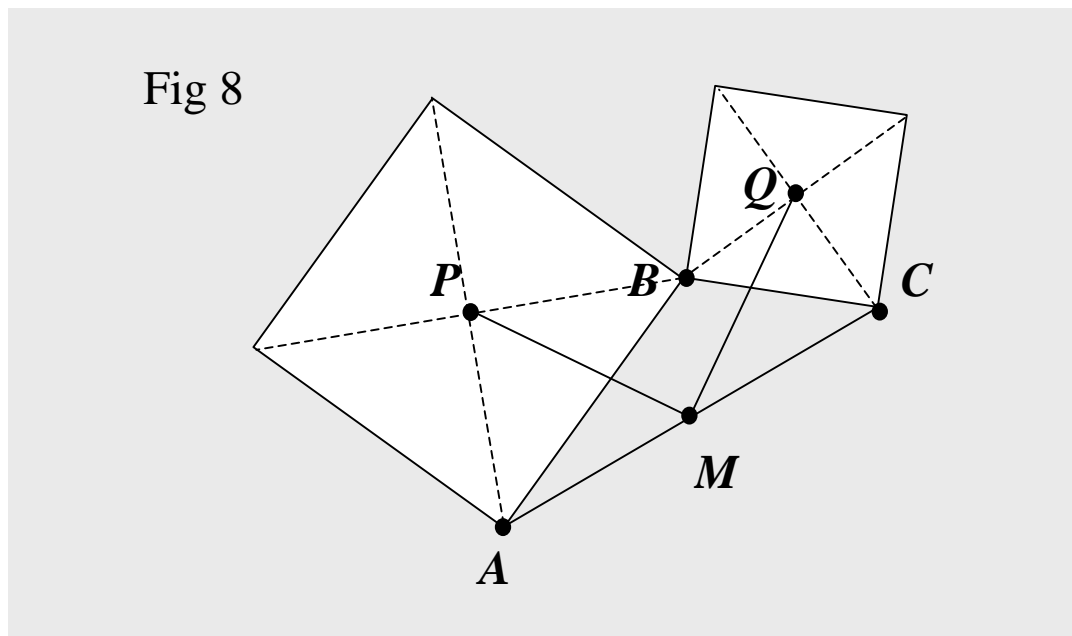
Why, though, the midpoint in the side facing the squares?

Well, that's the starter key, which opens the door to the solution, the proof in this case. And such a key requires a mathematical insight, which requires a lot of mental math, practices, and trial-and-error, along with creativity, and of course, strong foundation of geometry, robust basics.

The key is not the point here, though. The point is, you learn how math basics are used in problem solving processes.

So when basics are introduced or indicated during this lesson, you may want to take a good look at those closely, and try understanding why or how the basics are working in the construction of the proof, i.e., problem solving processes.

Let's now move on. Let's this time, as shown below, connect  $P$  and  $M$  with a line segment, and connect  $Q$  and  $M$  with another line segment.

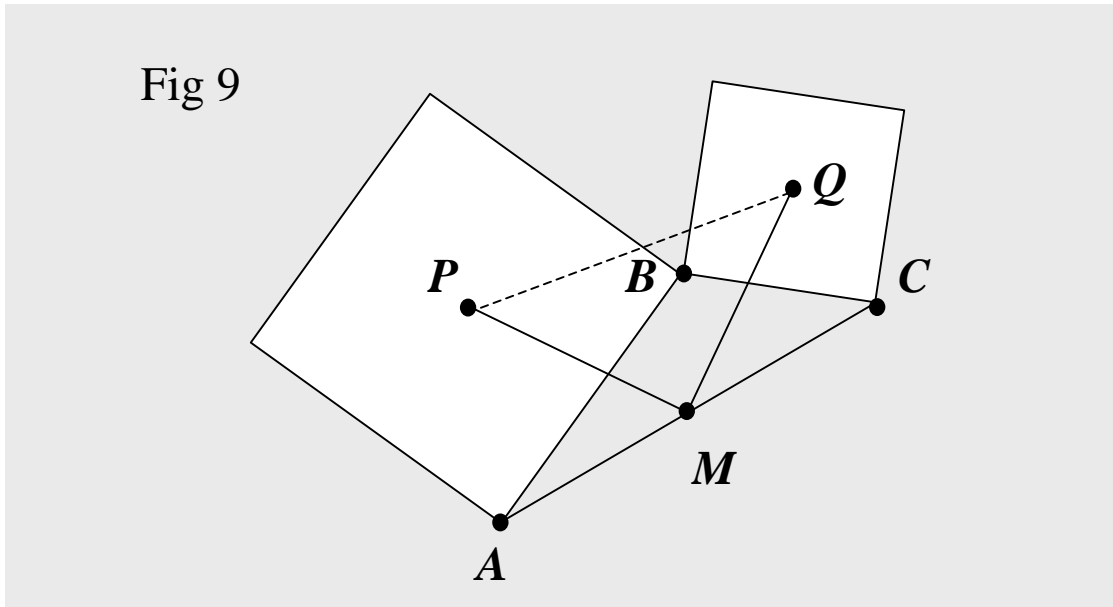


What then is the angle between  $PM$  and  $QM$ ?

- A.  $120^\circ$       B.  $90^\circ$       C.  $60^\circ$       D. inconsistent (that is, it depends on the squares and the triangle.)

It is a right angle, that is,  $90^\circ$ .

In other words, we get this:  $\angle PMQ = 90^\circ$ .



So if connecting the two centers and the midpoint, we get a right triangle, that is,  $\Delta PMQ$  is a right triangle. And that's not it. It's isosceles, too. And the same is true of any other triangle used as  $\Delta ABC$ , also.

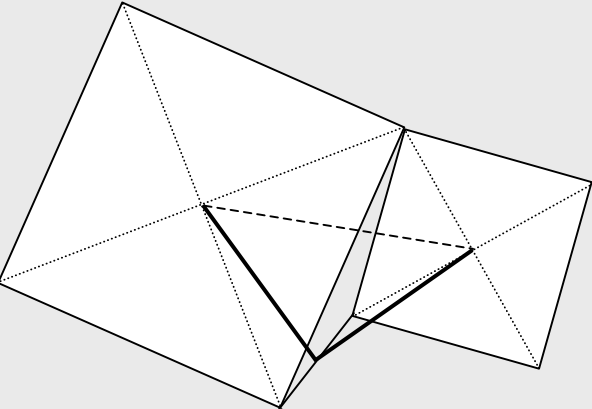
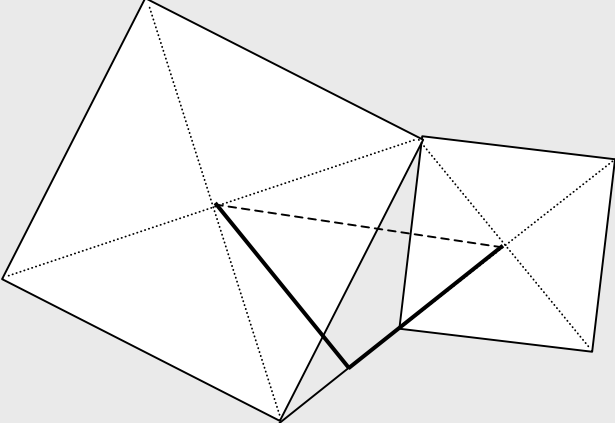
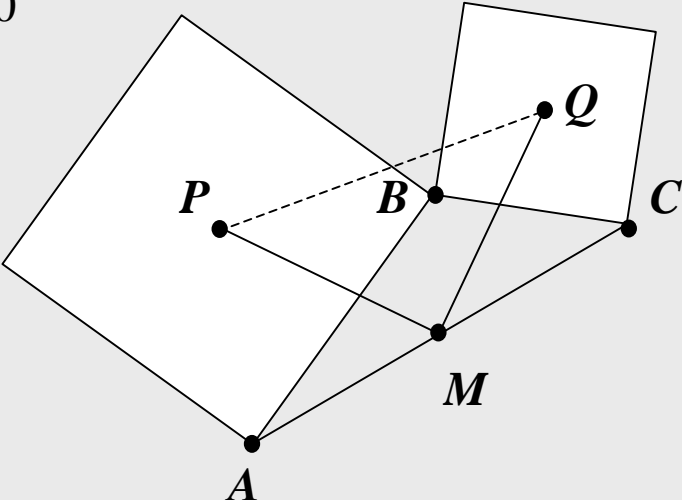
That is to say that whatever  $\Delta ABC$  may be,  $\Delta PMQ$  is a right triangle isosceles.

How do we know, though, if it is the case?

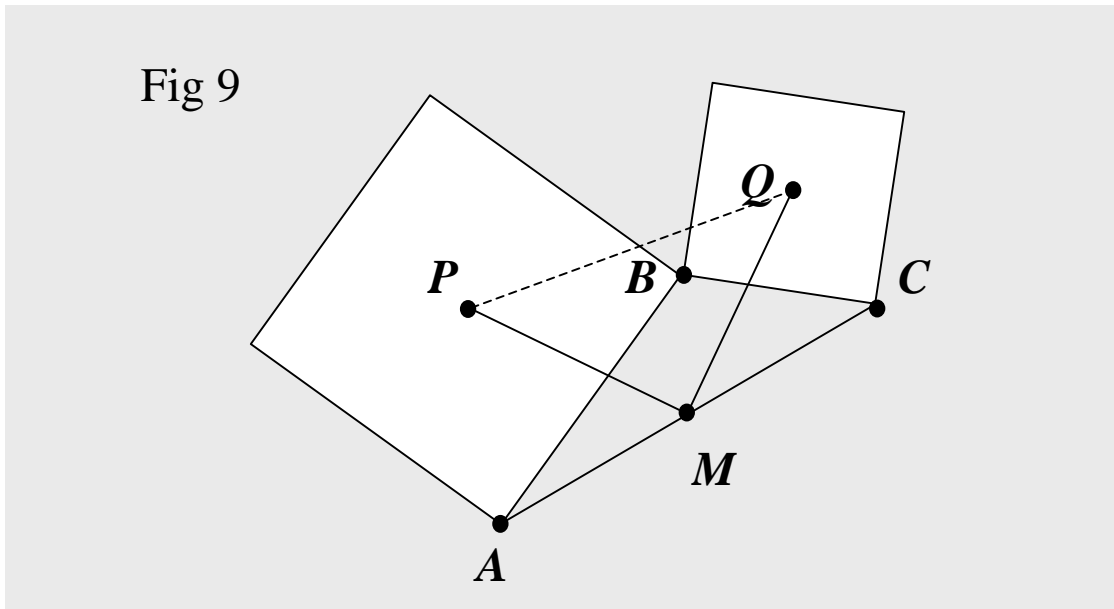
Let's first, try some other triangles, and see what happens,

It seems to be the case, but we need the proof.

Fig 10



So it can be another theorem if proved.



Well, thus, again, if connecting the two centers and the midpoint, we get a right triangle, that is,  $\Delta PMQ$  is a right triangle. And that's not it. It's isosceles, too. And the same is true of any other triangle used as  $\Delta ABC$ , also.

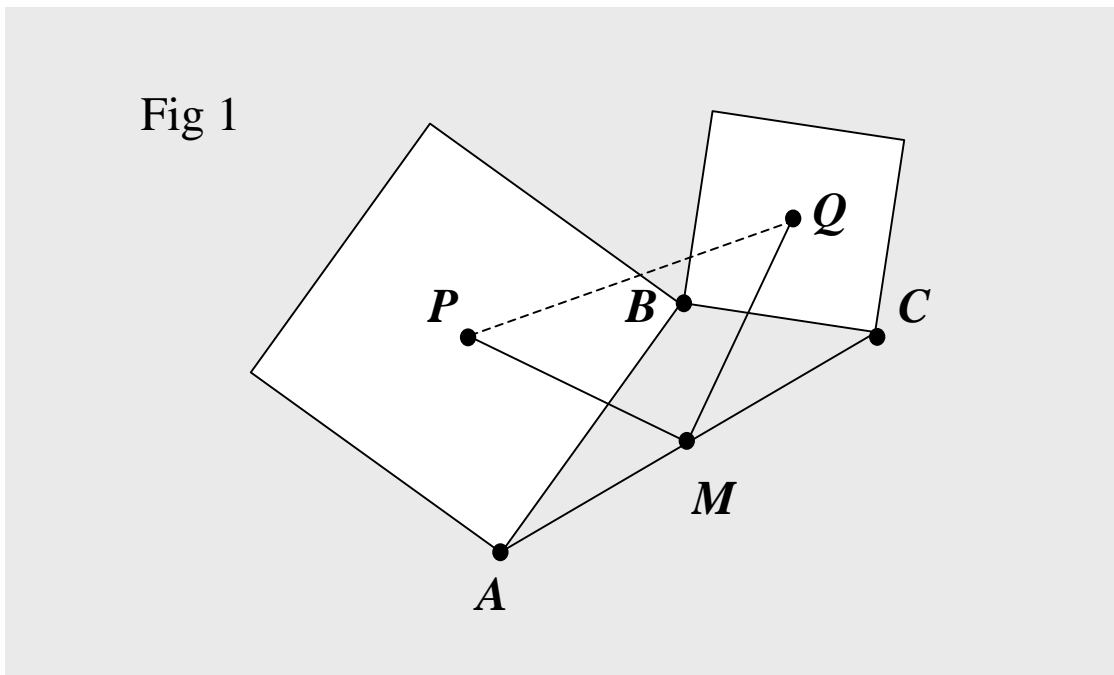
That is to say that whatever  $\Delta ABC$  may be,  $\Delta PMQ$  is a right triangle isosceles.

We'll do the proof in the next lesson.



## Van Aubel's Theorem Part 2

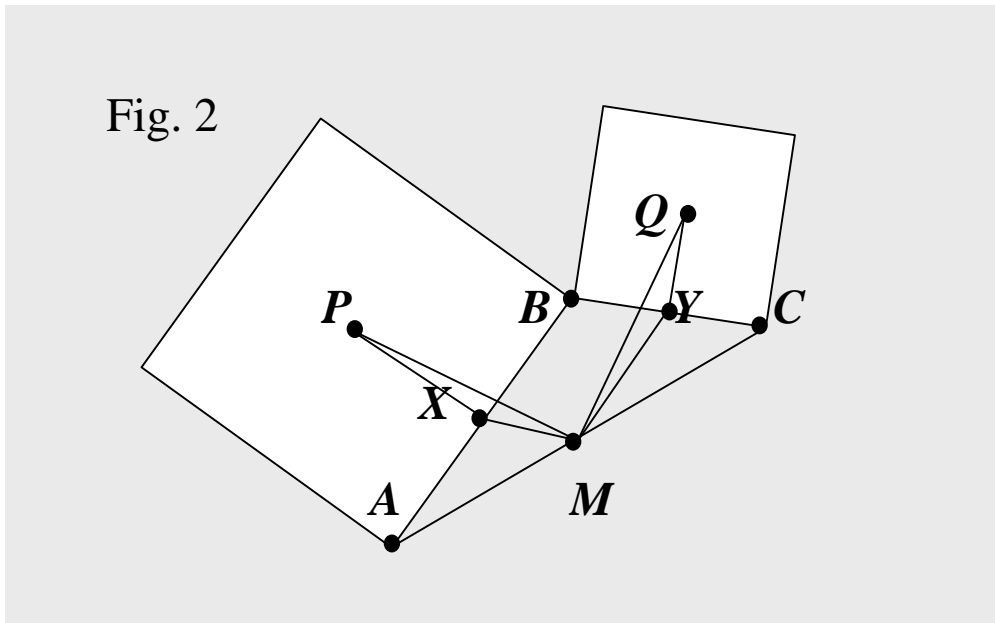
In the figure below, if connecting the two centers  $P$  and  $Q$  and the midpoint  $M$ , we get a right triangle, that is,  $\triangle PMQ$  is a right triangle. And that's not it. It's isosceles, too. And the same is true of any other triangle used as  $\triangle ABC$ , too.



That is to say that whatever  $\triangle ABC$  may be,  $\triangle PMQ$  is a right triangle isosceles. So it can be another theorem if proved.

Let's see now how we can prove it.

Suppose first, in the figure below,  $X$  is the midpoint of  $AB$ , and  $Y$  is the midpoint of  $BC$ .



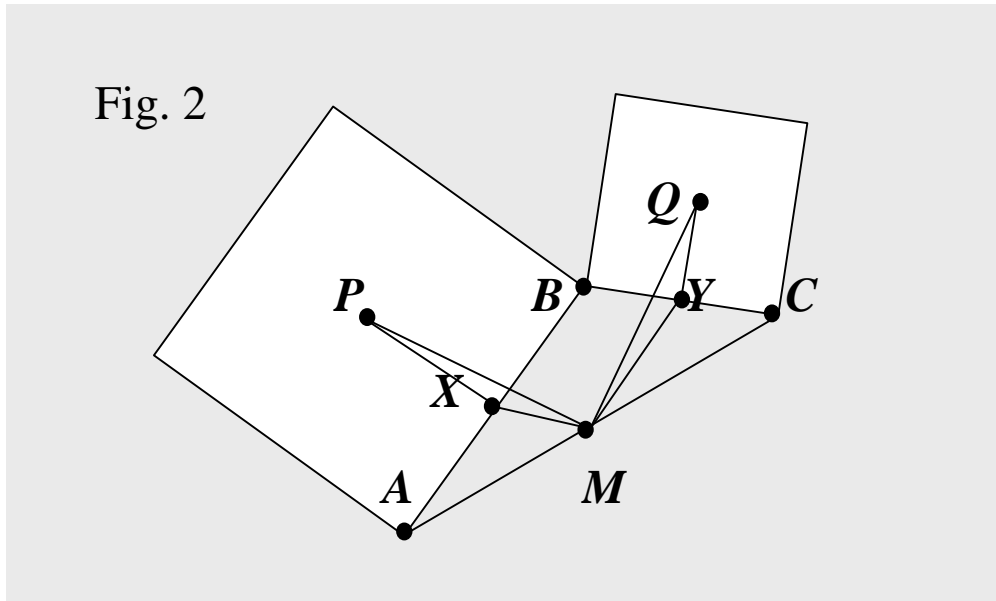
Then, since the two points  $P$  and  $Q$  are the centers of the squares, we get these:

$$PX \perp BX, \quad |PX| = |BX|, \quad QY \perp BY, \quad |QY| = |BY|$$

That is to say that the two line segments  $PX$  and  $BX$  meet at a right angle and have the same lengths, and the same is true of the two line segments  $QY$  and  $BY$ , too.

What then about  $YM$  and  $XM$ ?

We know the fact that  $X$ ,  $Y$ , and  $M$  are the midpoints of the three sides of the triangle  $ABC$ . So what can we get?



We can get this:  $\triangle MCY \sim \triangle ACB$ .

That is to say that  $\triangle MCY$  and  $\triangle ACB$  are similar.

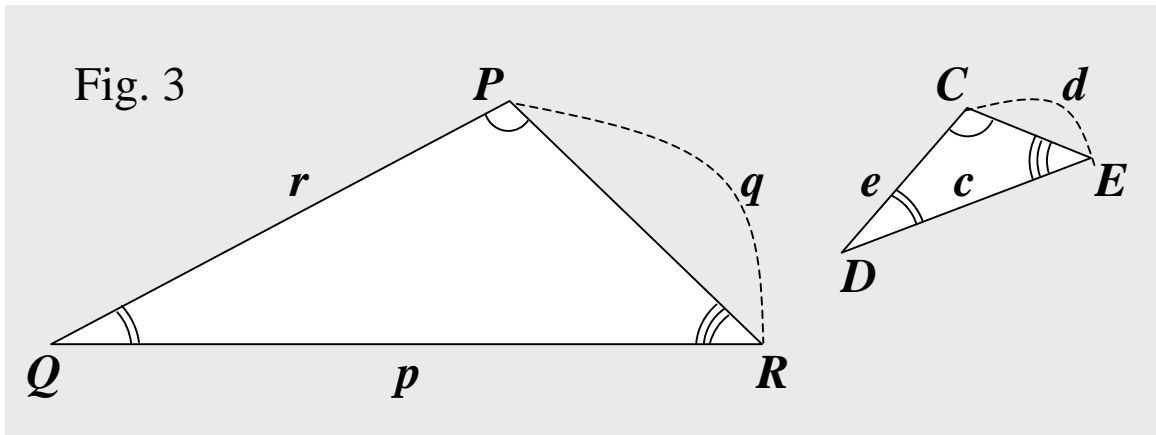
And using the fact that they are similar, we can get the information on  $YM$  and  $XM$ .

Why similar, though?

We will get it using tools called math basics.

And they are as follows.

We have math basics called **definitions for similar triangles**, which are in two groups. Each group is made of three versions, and one group is as follows.



$$(1) \quad \triangle PQR \sim \triangle CDE \Leftrightarrow \frac{c}{p} = \frac{d}{q} = \frac{e}{r}$$

$$(2) \quad \triangle PQR \sim \triangle CDE \Leftrightarrow \angle P = \angle C, \text{ and } \angle Q = \angle D$$

$$(3) \quad \triangle PQR \sim \triangle CDE \Leftrightarrow \angle P = \angle C, \text{ and } \frac{e}{r} = \frac{d}{q}$$

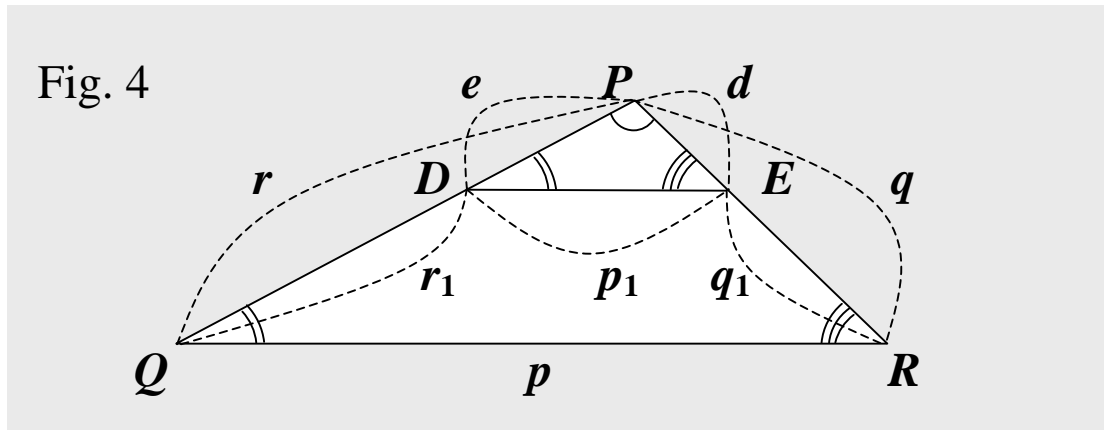
And if any of the three above is true, we get these:

$$(a) \quad \frac{c}{p} = \frac{d}{q} = \frac{e}{r}, \text{ three equal ratios, the same scale factor}$$

$$(b) \quad \angle P = \angle C, \angle Q = \angle D, \text{ and } \angle R = \angle E,$$

i.e., both triangles have the same angle groups.

And the other group is as follows. Note that  $\triangle PDE$  is nested inside  $\triangle PQR$ , and **very particularly** that  $\angle P$  is common.



$$(1-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \frac{d}{q} = \frac{e}{r}$$

$$(2-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \angle Q = \angle D$$

$$(3-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \frac{e}{r_1} = \frac{d}{q_1}$$

And if any of the three above is true, we get these:

$$(a-1) \quad \frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}, \text{ three equal ratios, the same scale factor}$$

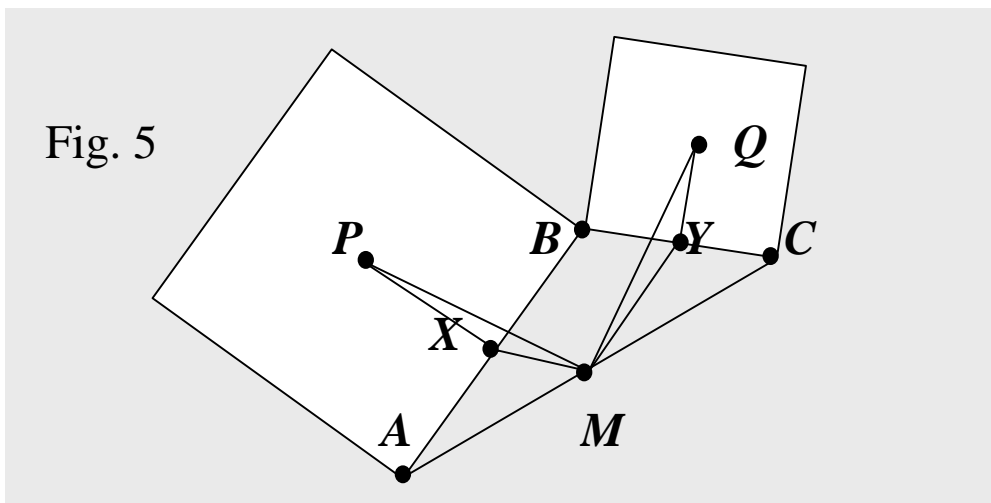
$$(b-1) \quad \angle Q = \angle D, \text{ and } \angle R = \angle E,$$

i.e., both triangles have the same angle groups.

If not sure of the basics above, refer to the lessons, **Similar Triangles**. They're often used in problem solving processes in many kinds. So better keep them running very well.

Now, answering the question, why similar, we can say this:

First,  $X$ ,  $Y$ , and  $M$  are the midpoints of the sides of  $\triangle ACB$ , so we can **expect** that  $\triangle MCY$  and  $\triangle ACB$  are similar, and can **check** if they are so using one of the definitions above.

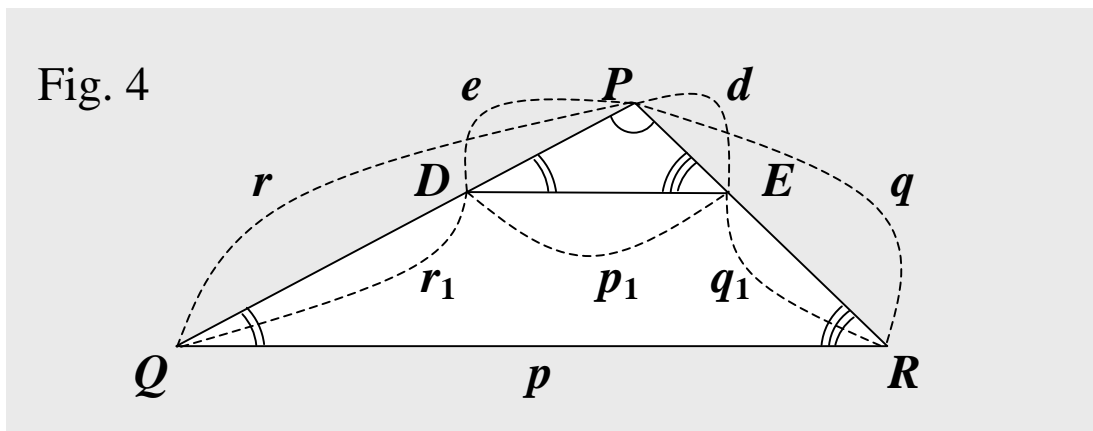
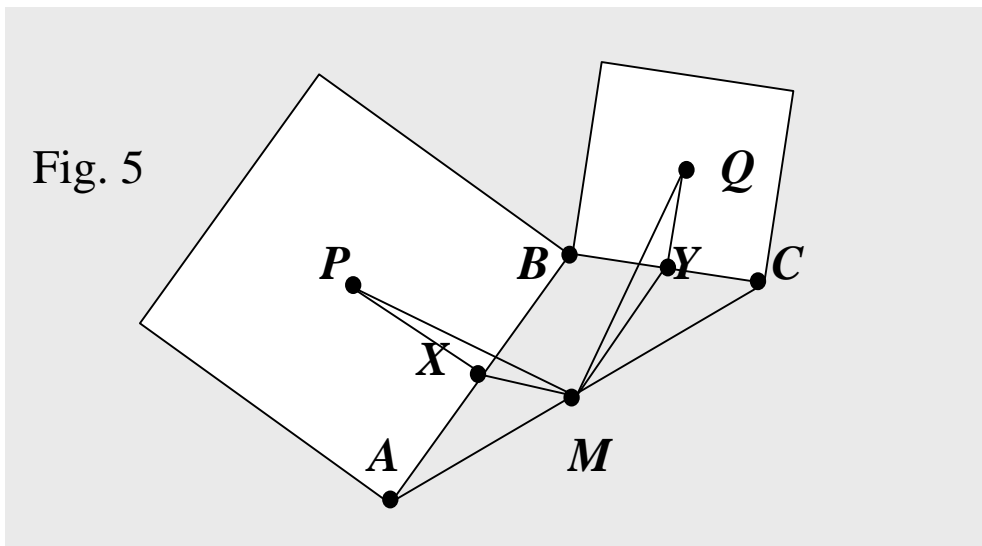


What then is the definition? We have two groups, and each is made of three versions. So which one do we use?

We know that  $\triangle MCY$  is nested inside  $\triangle ACB$ , and we have enough information for the ratios between the sides, so we can use the definition (1-1), shown in the next page, too.

And if it's true, we can use (a-1) below to see what  $YM$  is.

Note that  $\angle C$  is common to  $\triangle MCY$  and  $\triangle ACB$ , and that  $\triangle MCY$  matches  $\triangle PDE$ , and  $\triangle ACB$  matches  $\triangle PQR$ .



(a-1)  $\frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}$ , three equal ratios, the same scale factor

Checking now thus, to see if  $\triangle MCY \sim \triangle ACB$  using (1-1)

below, we get this:  $\frac{|CY|}{|CB|} = \frac{1}{2} = \frac{|CM|}{|CA|}$ , so they are similar.

Note that in Fig. 5,  $CY$  matches  $d$  in Fig 4,  $CB$  matches  $q$ ,  $CM$  matches  $e$ , and  $CA$  matches  $r$ .

Fig. 5

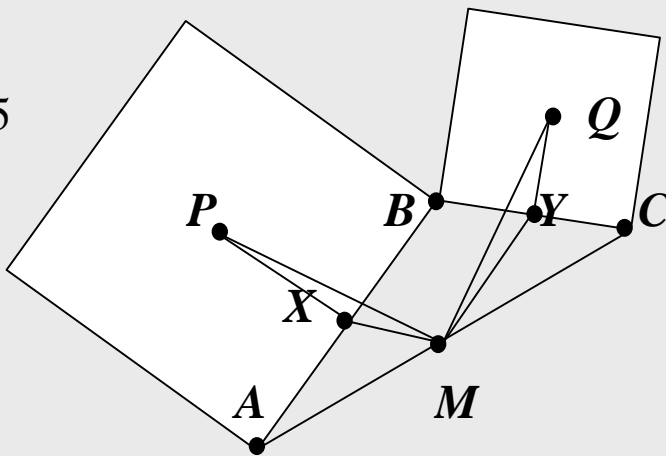
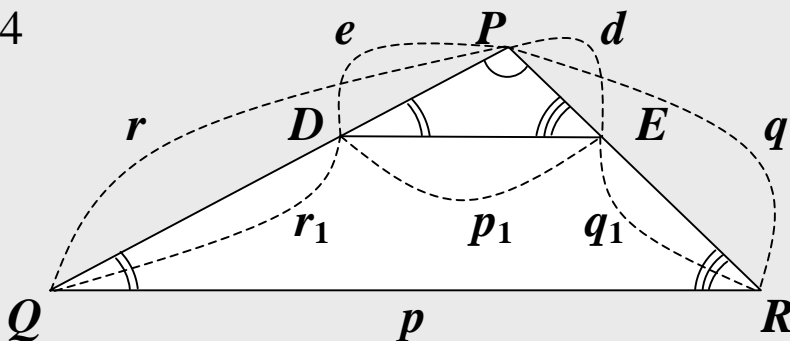


Fig. 4

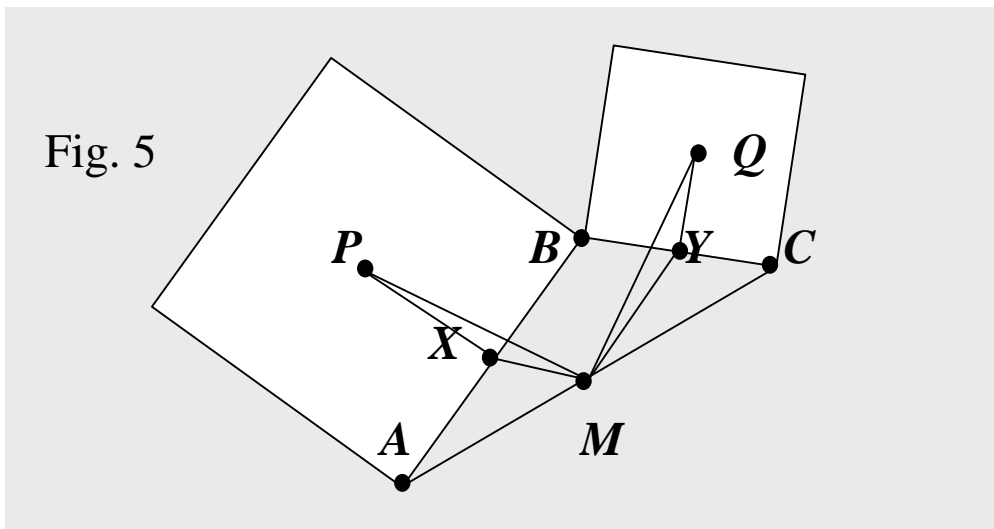


$$(1-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \frac{d}{q} = \frac{e}{r}$$

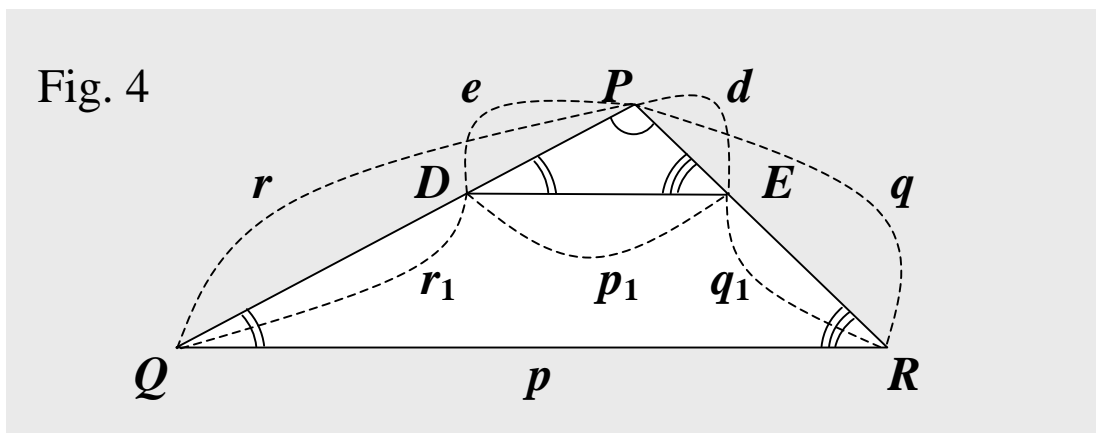
So using (a-1) where  $p_1$  matches  $YM$ , and  $p$  matches  $BA$ ,

we get this:  $\frac{|YM|}{|BA|} = \frac{1}{2}$ , so we get this:  $|YM| = \frac{|BA|}{2}$ ,

which is  $|BX|$ . Thus, we get this:  $|YM| = |BX|$ .



(a-1)  $\frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}$ , three equal ratios, the same scale factor

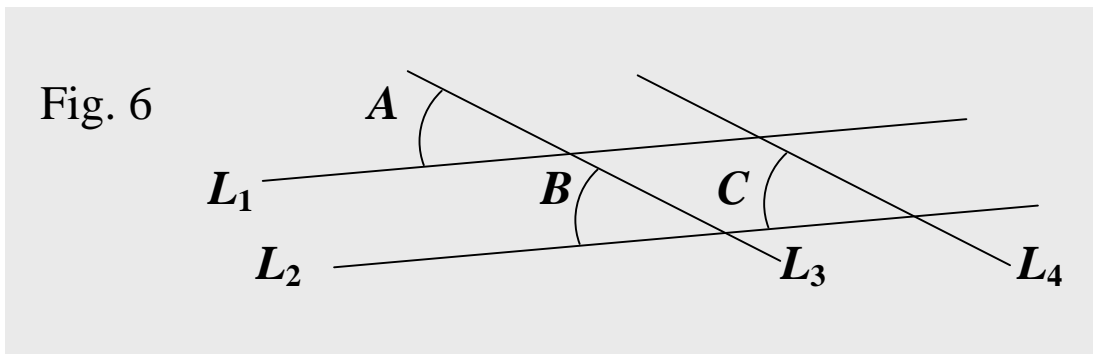


And we get this, too:  $YM \parallel BX$ . Why parallel, though?

Answering the question, why parallel, we can say this:

We have a **basic** that if corresponding angles are equal, the lines crossed by the transversal are parallel, and vice versa.

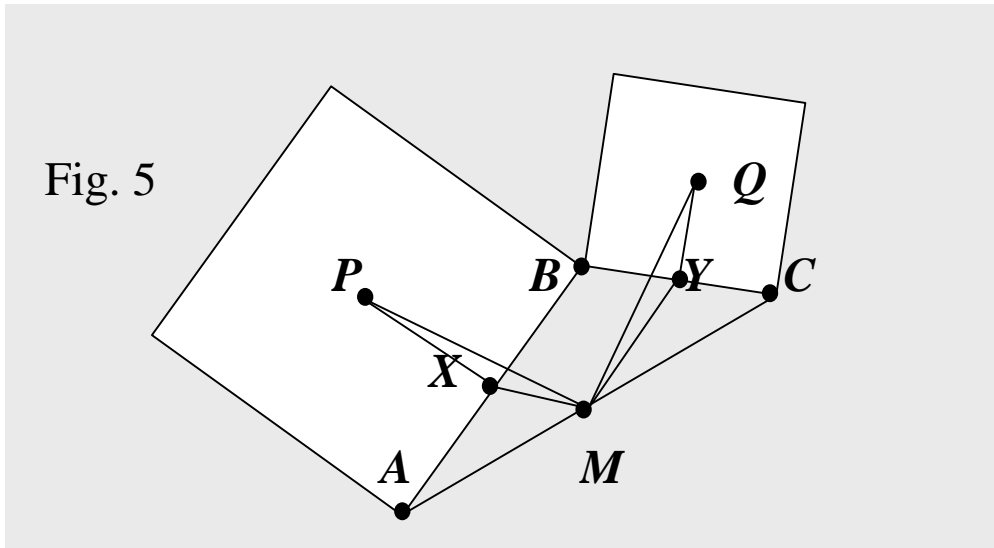
So now, in Fig. 6 below, if we have this:  $\angle A = \angle B$ , we get this:  $L_1 \parallel L_2$ , because  $\angle A$  and  $\angle B$  are corresponding angles, since  $L_3$  is the transversal.



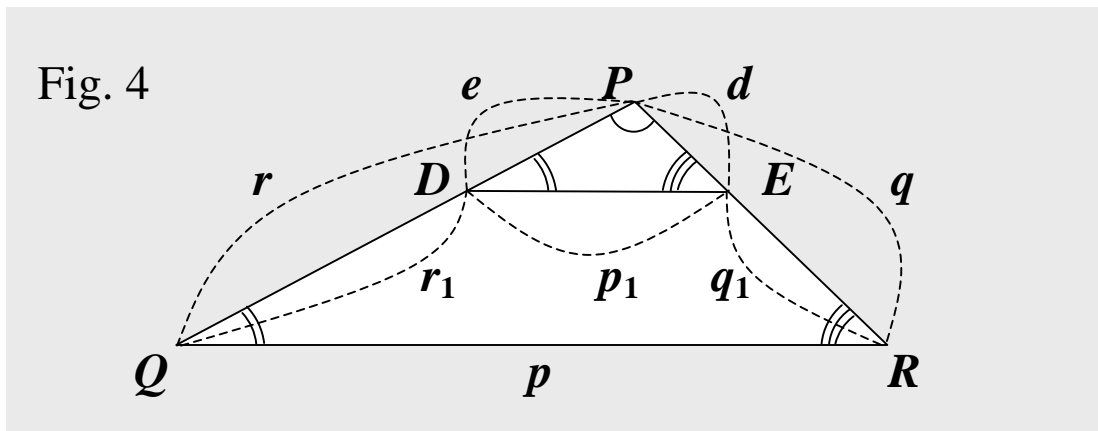
So again, if corresponding angles are equal, the two lines with the corresponding angles are parallel, and vice versa.

We can use the basic above solving many problems. If not sure of it, refer to the lessons called **Angles and Lines**.

And now, we know that  $\triangle MCY$  and  $\triangle ACB$  are similar.



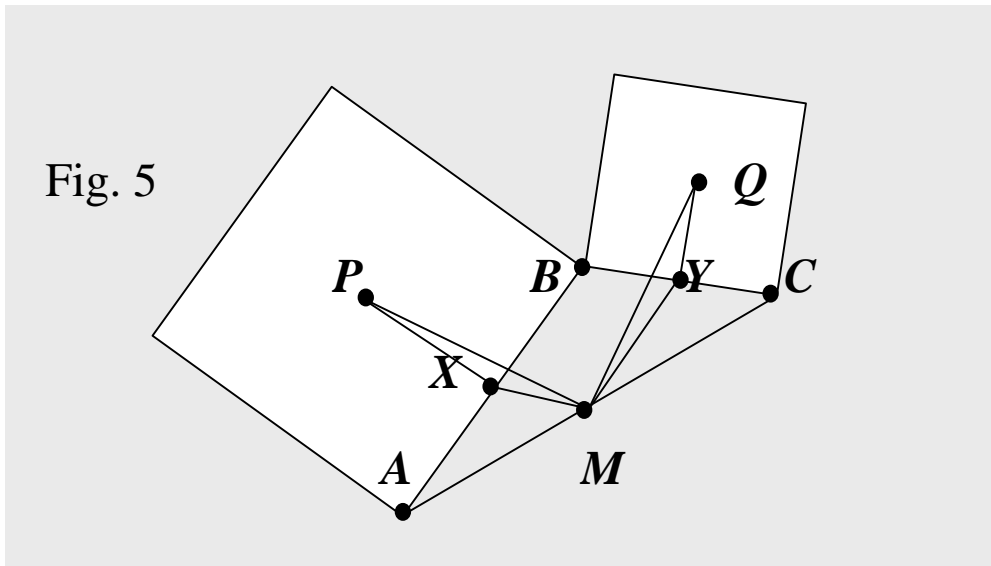
So both triangles have the same angle groups.  
And that's what the definition (2-1) below is saying.



$$(2-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \angle Q = \angle D$$

Note that  $\angle Q$  matches  $\angle CAB$ , and  $\angle D$  matches  $\angle CMY$ .

Thus,  $\angle CMY = \angle CAB$ , and these two angles are the same corresponding angles if we take  $AC$  as the transversal.



So we get this:  $YM \parallel BA$ , since the two angles above are the same corresponding angles.

And  $BX$  is a part of  $BA$ , so, we get this:  $YM \parallel BX$ .

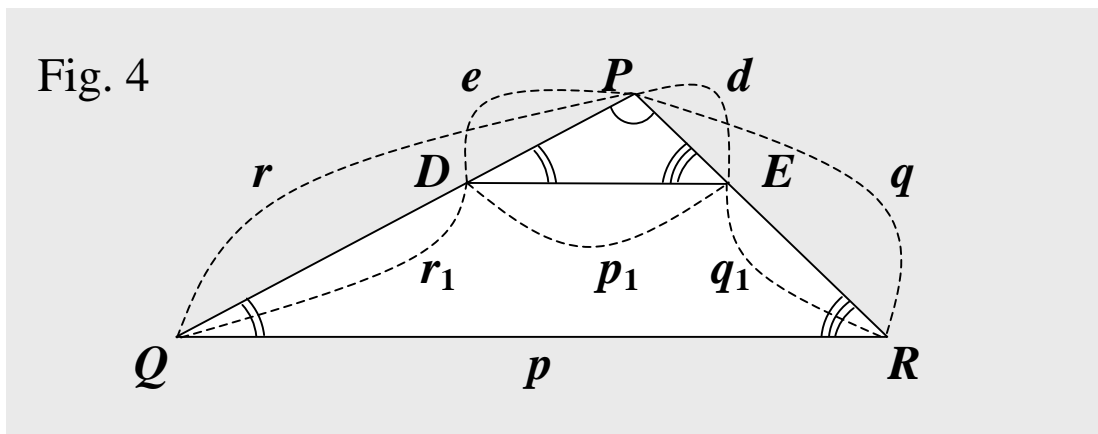
By the same token, we get this:  $|XM| = |BY|$ , too.

And we get this:  $XM \parallel BY$ , also. Why, though?

It's because of the same basics covered above. As a practice though, we may want to go over the basics, since the basics matter in these lessons. Basics matter anyway, though. So let's go them over now.

Before we start afresh to go them over, let me point this out:

If you get to see this kind of situation where **two triangles are nested and similar**, you can directly use a fact that **the two sides away from each other are parallel**, as in the case of the two sides  $p_1$  and  $p$  in the figure below.



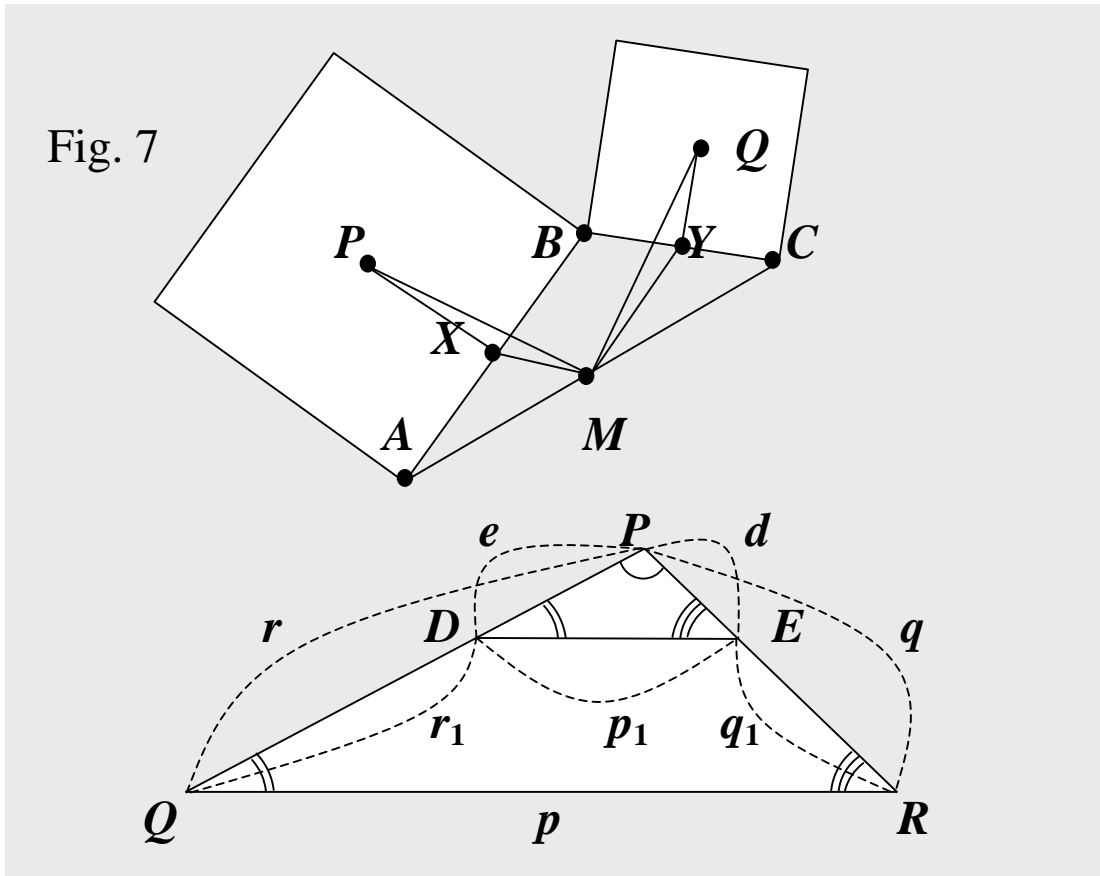
$$(2-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \angle Q = \angle D$$

In the figure above, we have this:  $\angle Q = \angle D$ , which means that corresponding angles are equal, since  $\angle Q$  and  $\angle D$  are corresponding angles. And we've covered the fact that if corresponding angles are equal, the two lines with the corresponding angles are parallel, and vice versa.

So we get this:  $p_1 \parallel p$ , that is,  $p_1$  and  $p$  are parallel.

Let's now start going over the basics.

In this case,  $\Delta AXM$  is nested inside  $\Delta ABC$ , and matches  $\Delta PDE$ . And  $\Delta ABC$  matches  $\Delta PQR$ . This time, let's use the definition (3-1), and then, use (a-1) to see what  $XM$  is.



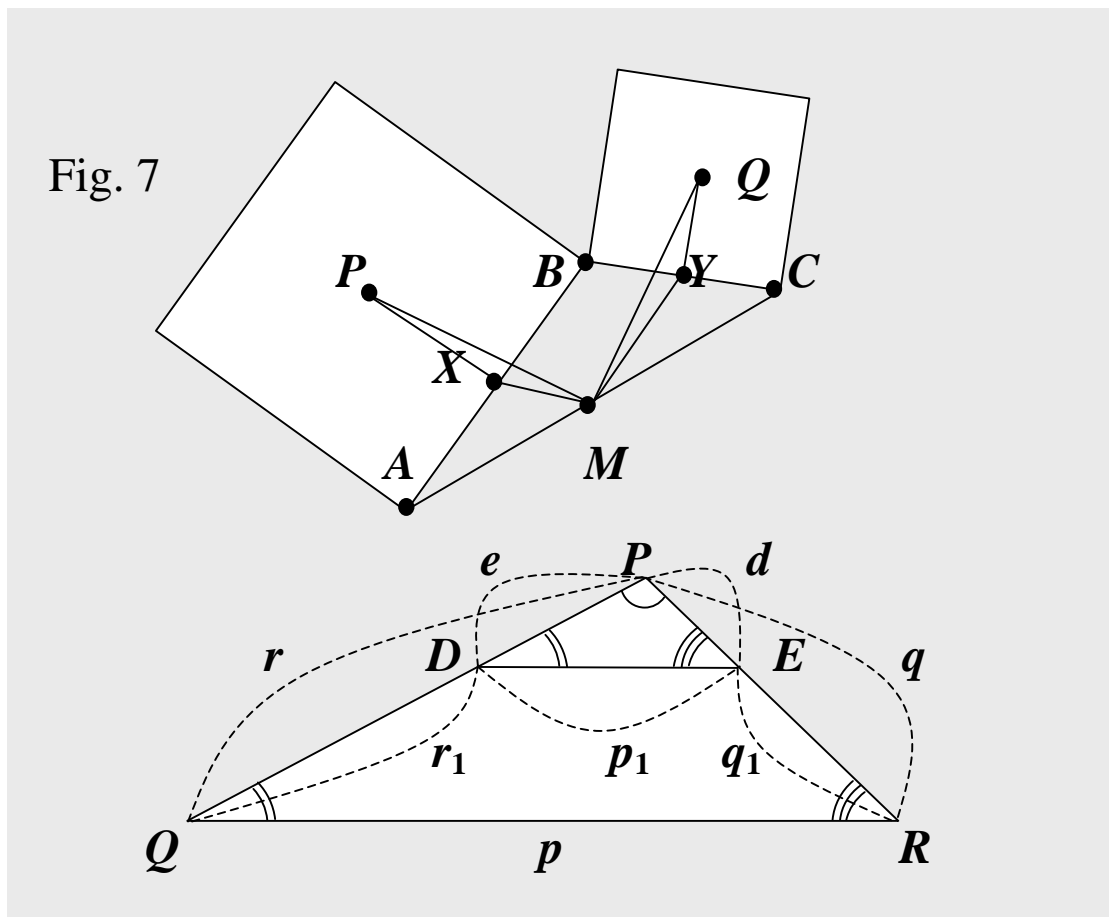
$$(3-1) \quad \Delta PQR \sim \Delta PDE \Leftrightarrow \frac{e}{r_1} = \frac{d}{q_1} \text{ where } e \text{ matches } r_1$$

$AX$ ,  $r_1$  matches  $XB$ ,  $d$  matches  $AM$ , and  $q_1$  matches  $MC$ .

$$(a-1) \quad \frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}, \text{ three equal ratios, the same scale factor}$$

So now again, the three points  $X$ ,  $Y$ , and  $M$  are the midpoints of the three sides of  $\triangle ABC$ .

And this time, the two triangles  $\triangle AXM$  and  $\triangle ABC$  share  $\angle A$ . So we can **expect** that  $\triangle AXM$  and  $\triangle ABC$  are similar.



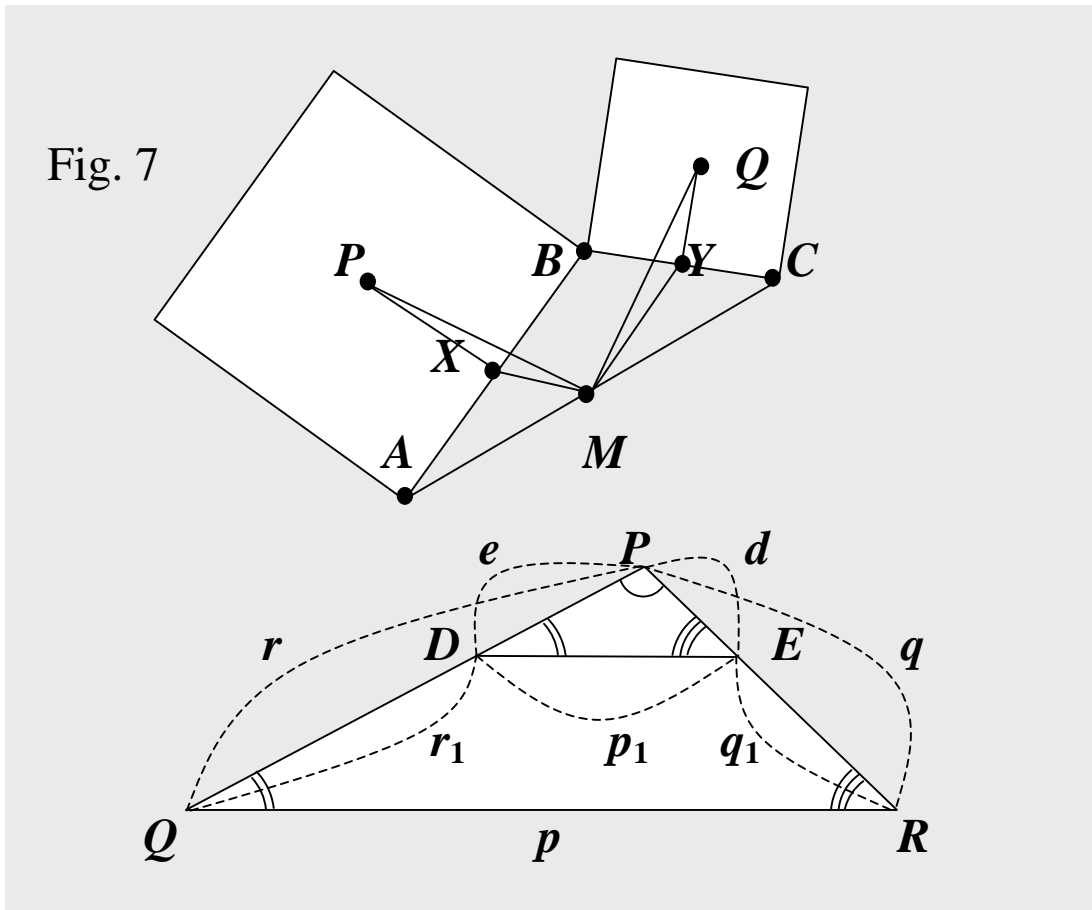
$$(3-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \frac{e}{r_1} = \frac{d}{q_1} \text{ where } e \text{ matches } r_1$$

$AX$ ,  $r_1$  matches  $XB$ ,  $d$  matches  $AM$ , and  $q_1$  matches  $MC$ .

Checking thus, to see if  $\triangle AXM$  and  $\triangle ABC$  are similar using

(3-1), we get this:  $\frac{|AX|}{|XB|} = \frac{1}{1} = \frac{|AM|}{|MC|}$ , which shows that

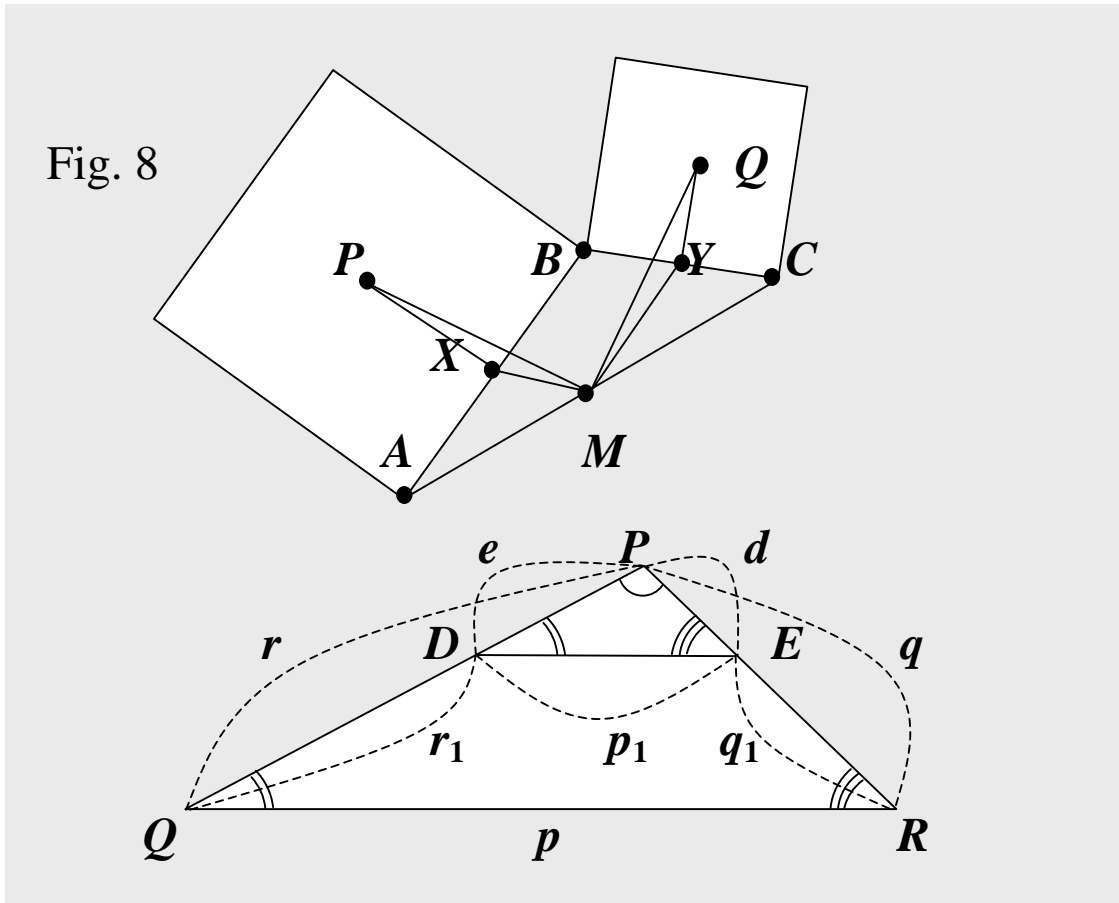
they are indeed similar, since  $\angle A$  is common.



$$(3-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \frac{e}{r_1} = \frac{d}{q_1} \text{ where } e \text{ matches}$$

$AX$ ,  $r_1$  matches  $XB$ ,  $d$  matches  $AM$ , and  $q_1$  matches  $MC$ .

And therefore, using (a-1), we get this:  $\frac{|AM|}{|AC|} = \frac{1}{2} = \frac{|AX|}{|AB|}$ .

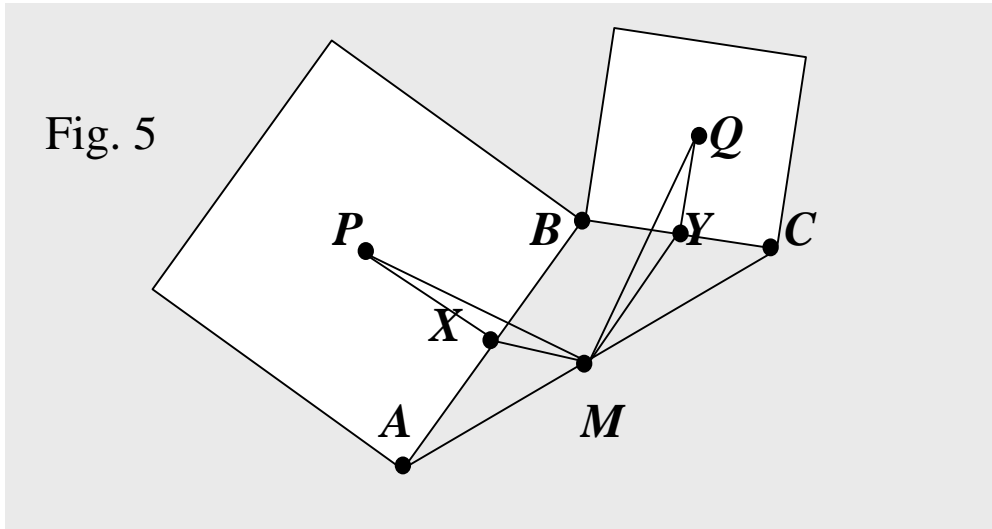


(a-1)  $\frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}$ , equal ratios, the same scale factor

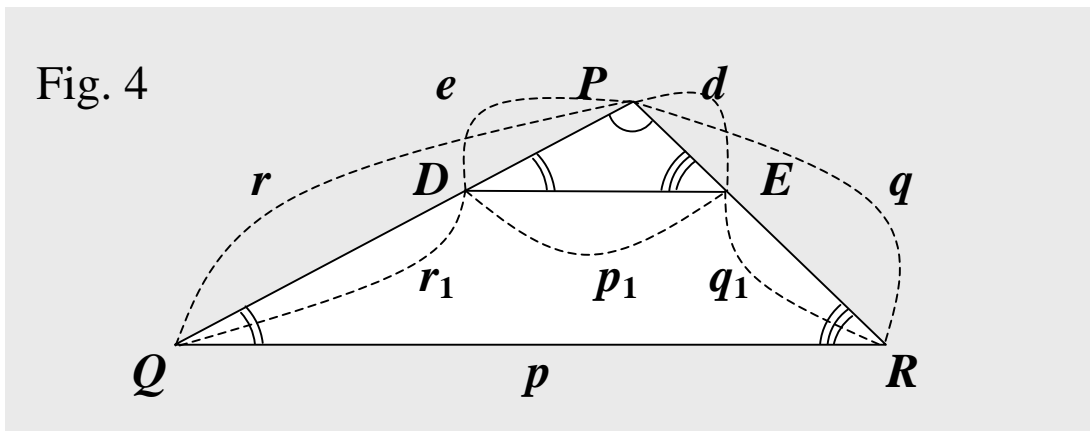
So we get this:  $\frac{|XM|}{|BC|} = \frac{1}{2}$ .

Now that we have this:  $\frac{|XM|}{|BC|} = \frac{1}{2}$ , we can get this:

$|XM| = \frac{|BC|}{2}$ , which is  $|BY|$ . So we get this:  $|XM| = |BY|$ .



And, of course, we get this, too:  $XM \parallel BY$ , because the two triangles  $\triangle AMX$  and  $\triangle ACB$  are similar. In the figure below, we have this:  $\triangle PDE \sim \triangle PQR$ , so we have this, too:  $p_1 \parallel p$ .

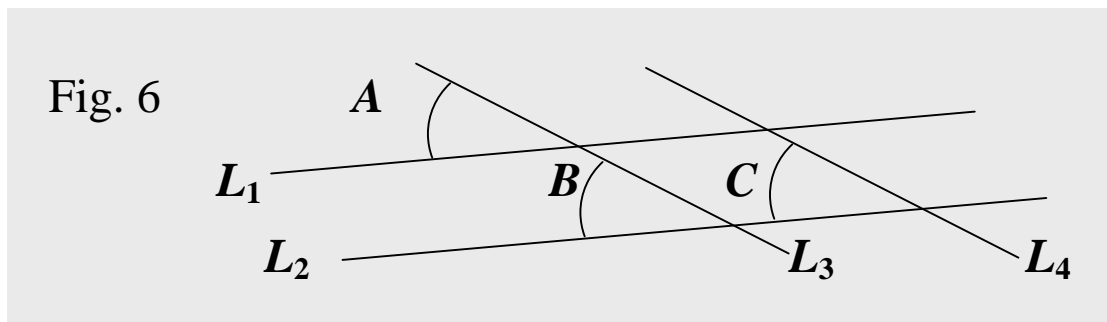


If however, answering again, the question, why parallel, we can say this again:

If the two lines with corresponding angles are parallel, the corresponding angles are equal, and vice versa.

So if corresponding angles are equal, the two lines with the corresponding angles are parallel.

Now, therefore, in Fig. 6 below, if we have this:  $\angle A = \angle B$ , we get this:  $L_1 \parallel L_2$ , because the two angles  $\angle A$  and  $\angle B$  are corresponding angles, since  $L_3$  is the transversal.

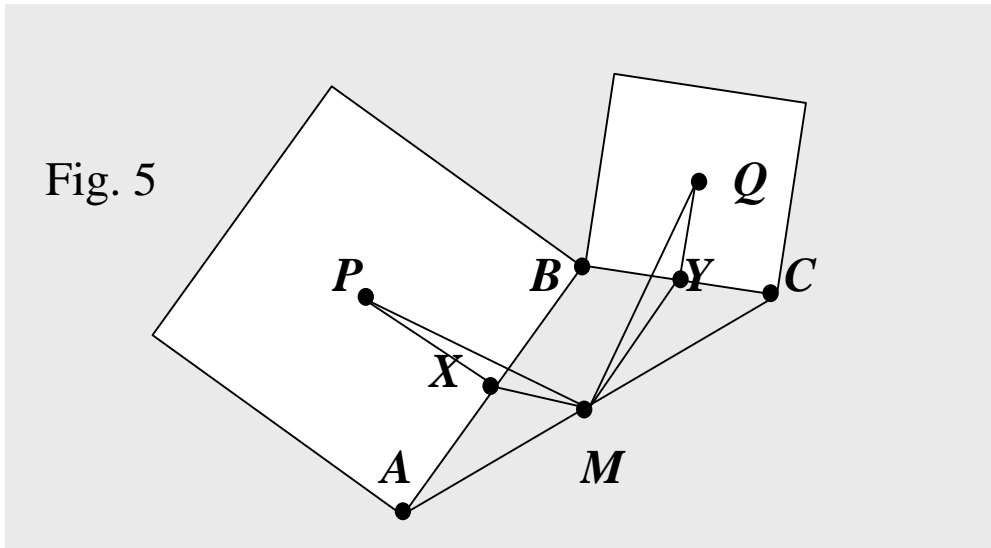


So again, if corresponding angles are equal, the two lines with the corresponding angles are parallel, and vice versa.

And we often use the basic above solving many problems.

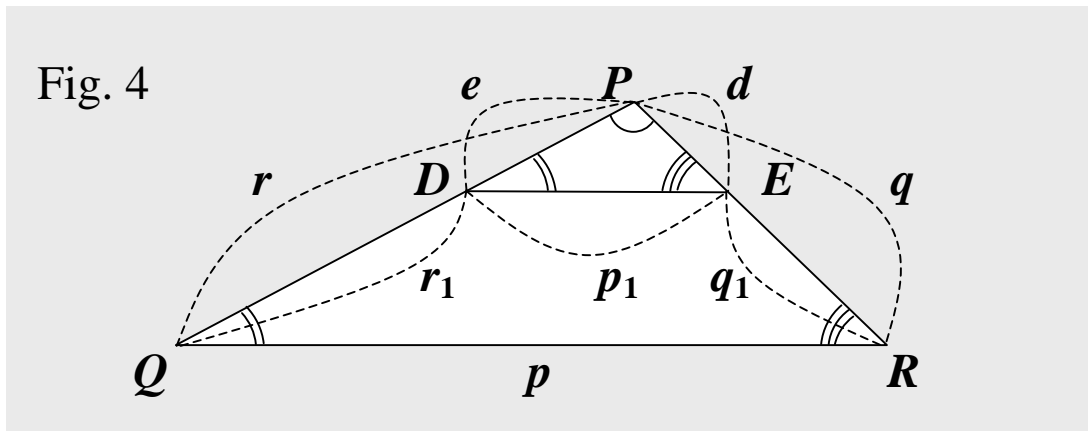
So you may want know and keep it very well.

And now, we know that  $\triangle AMX$  and  $\triangle ACB$  are similar.



So both triangles have the same angle groups.

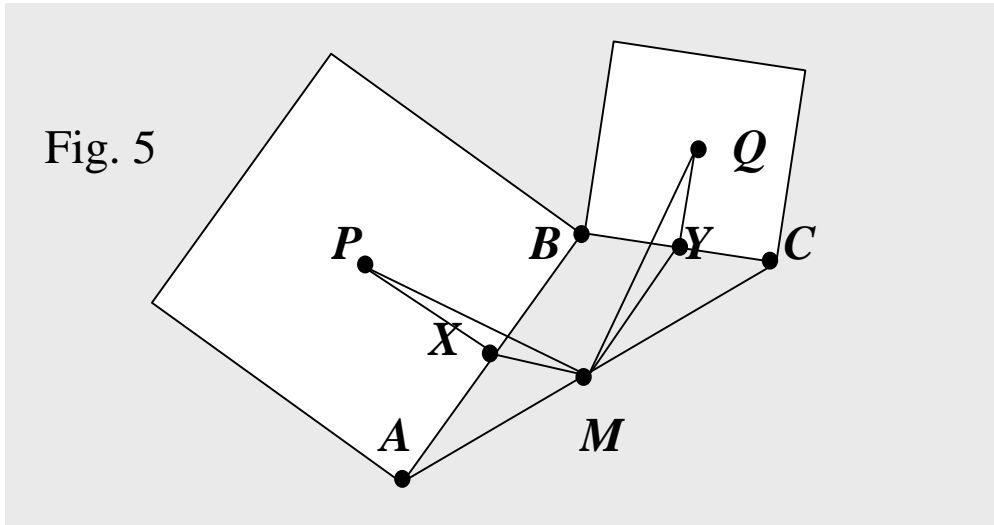
And that's what the definition (2-1) below is saying.



$$(2-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \angle Q = \angle D$$

Note that  $\angle Q$  matches  $\angle ACB$ , and  $\angle D$  matches  $\angle AMX$ .

Thus,  $\angle AMX = \angle ACB$ , and the two angles are the same corresponding angles if we take  $AC$  as the transversal,.



So we get this:  $XM \parallel BC$ , since the two angles above are the same corresponding angles.

And  $BY$  is a part of  $BC$ , so, we get this:  $XM \parallel BY$ .

And summing up, for now, we've got these:

$$|YM| = |BX|, \quad YM \parallel BX, \quad |XM| = |BY|, \quad XM \parallel BY$$

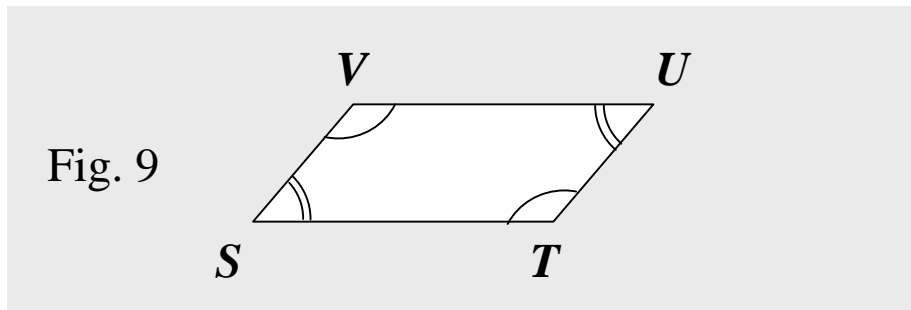
So the opposite sides are of the same length and parallel.

What then is the tetragon  $XYMB$ ?

It's a parallelogram, and its definition can be as follows.

A tetragon is a parallelogram if and only if each pair of opposite sides are parallel.

What then about the same lengths?



We can make another definition using the same lengths the way as follows.

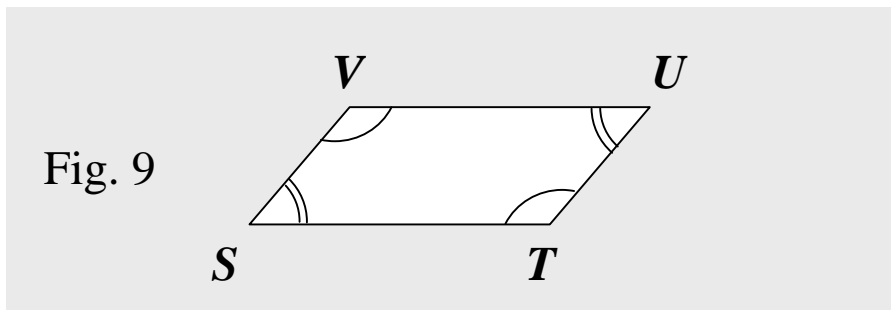
A tetragon is a parallelogram if and only if each pair of opposite sides have the same lengths.

And the conditions in the definitions are called properties of a parallelogram, too. So the properties are as follows.

Each pair of opposite sides are parallel.

And the two sides in each pair have the same lengths.

Solving thus, problems with a parallelogram, we can use the properties above. And that's not it. For example, each pair of angles facing each other are the same, and every pair of angles next to each other add up to  $180^\circ$ .



So in Fig. 16 above, we get these:  $\angle S = \angle U$  and  $\angle V = \angle T$ , and also, this:

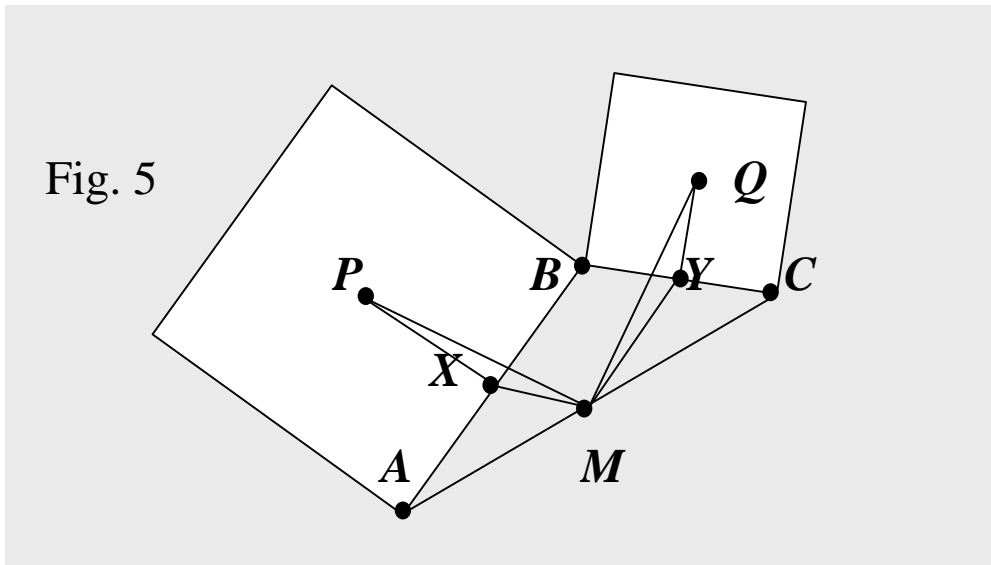
$$\angle S + \angle T = \angle T + \angle U = \angle U + \angle V = \angle V + \angle S = 180^\circ.$$

It has other properties, too. And we have a lot to say about parallelograms. So much for this story, though.

We may not want to go too far away from the proof we are now working on. If not quite sure of the concept of a parallelogram, you may want to refer to **Parallelograms** in the lessons, **Triangle Basics**.

So where are we now?

Been taking steps to prove that  $\triangle PMQ$  is an isosceles right triangle. We are not there yet. And we are now looking at a tetragon  $XMYB$ , which has turned out to be a parallelogram.

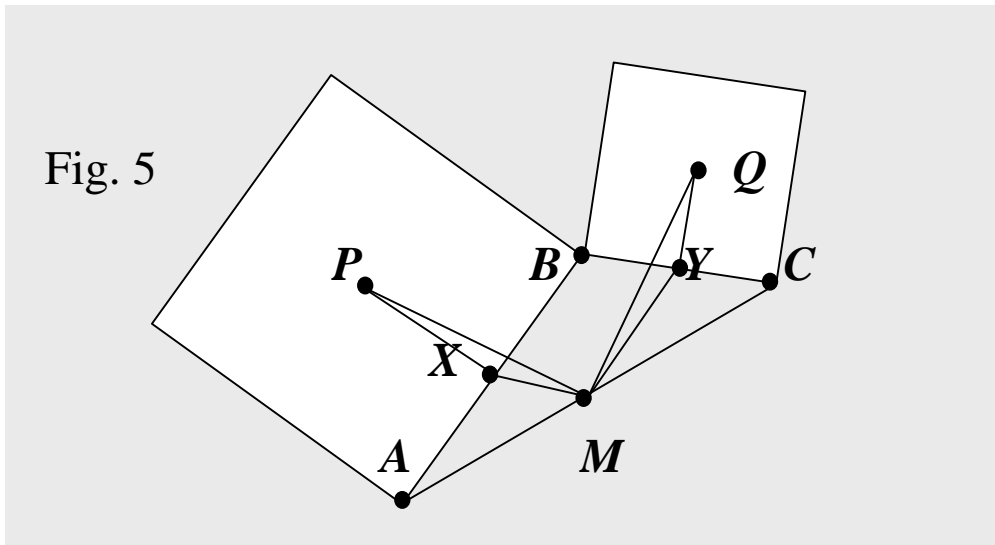


And we've covered a lot.

So let's now recapitulate what's been covered, so that we can see what we can work with and what could be done in the next step.

First, since the two points  $P$  and  $Q$  are the centers of the squares, we've got these:

$$PX \perp BX, \quad |PX| = |BX|, \quad QY \perp BY, \quad |QY| = |BY|$$



Next, the tetragon  $XMYB$  turned out to be a parallelogram, which means, we've got these, too:

$$|YM| = |BX|, \quad YM \parallel BX, \quad |XM| = |BY|, \quad XM \parallel BY$$

So now, putting things together, we get these:

$$PX \perp BX, \quad |PX| = |BX|, \quad QY \perp BY, \quad |QY| = |BY|$$

$$|YM| = |BX|, \quad YM \parallel BX, \quad |XM| = |BY|, \quad XM \parallel BY$$

And simplifying some of these:

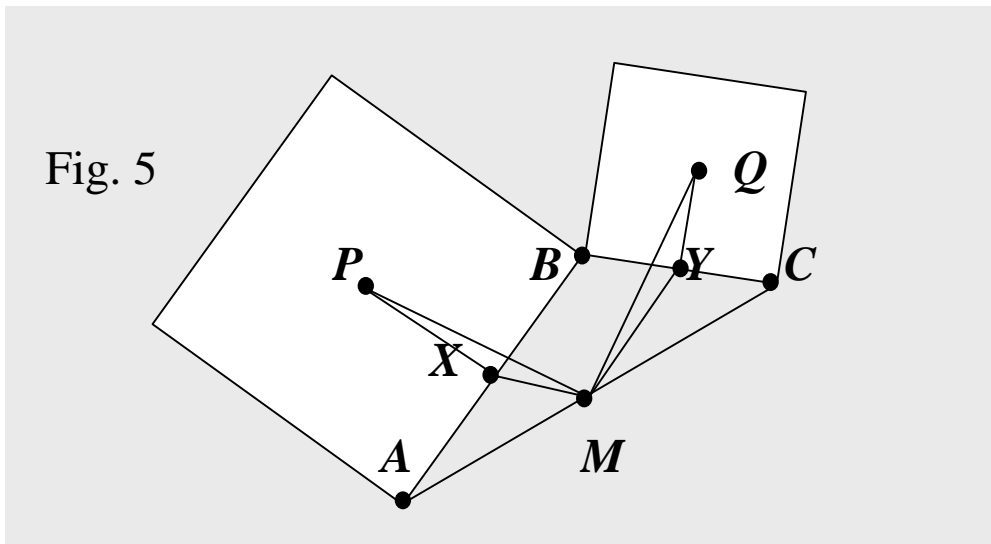
$$PX \perp BX, \quad |PX| = |BX|, \quad QY \perp BY, \quad |QY| = |BY|$$

$$|YM| = |BX|, \quad YM \parallel BX, \quad |XM| = |BY|, \quad XM \parallel BY$$

we can get these:

$$PX \perp BX, \quad QY \perp BY, \quad YM \parallel BX, \quad XM \parallel BY,$$

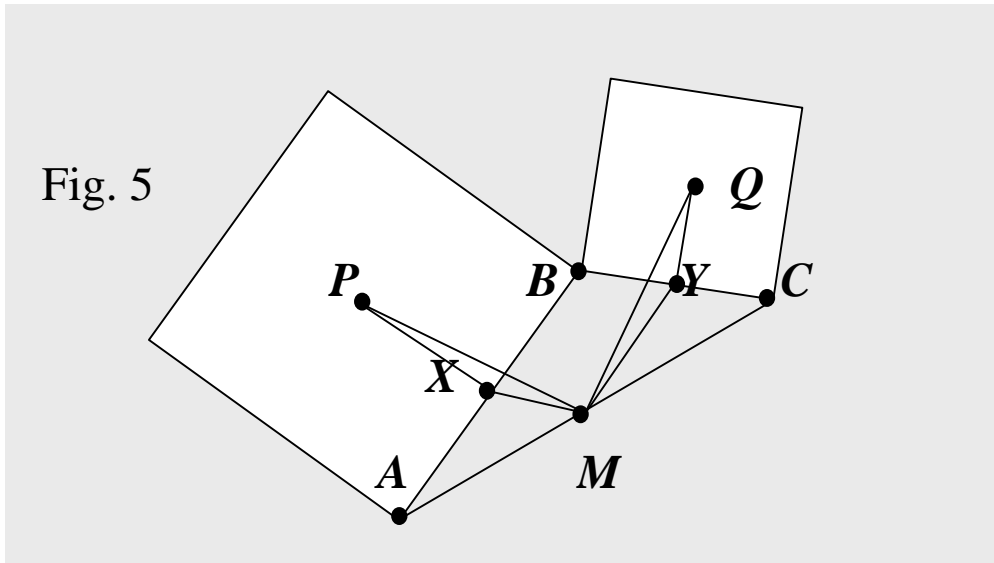
$$|PX| = |BX| = |YM|, \quad |QY| = |BY| = |XM|$$



Looking at the two triangles  $\triangle PXM$  and  $\triangle QYM$ , together with these two:  $|PX| = |BX| = |YM|$  and  $|QY| = |BY| = |XM|$ , we can notice that  $\triangle PXM$  and  $\triangle QYM$  share two sides.

What then are the two sides?

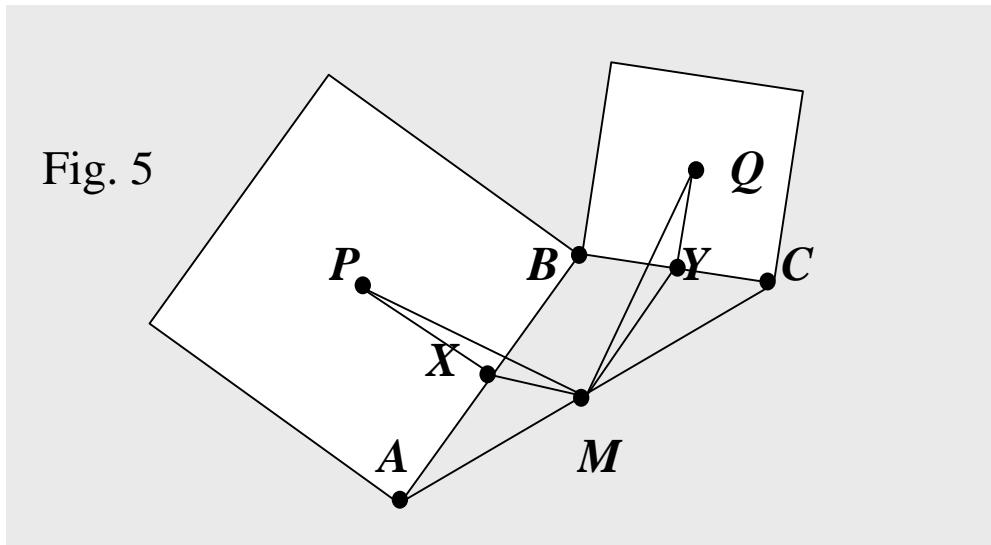
We have these:  $|PX| = |BX| = |YM|$  and  $|QY| = |BY| = |XM|$ ,  
 so we can get these two:  $|PX| = |YM|$  and  $|QY| = |XM|$ .



That is to say that in  $\Delta PXM$  and  $\Delta QYM$ , the side  $PX$  is the same as the side  $YM$ , and  $QY$  is the same as  $XM$ .

So what can we expect?

We can expect that  $\Delta PXM$  and  $\Delta QYM$  are identical; that is, we can expect this:  $\Delta PXM \equiv \Delta QYM$ , which is now thus, a hypothesis to be proven.



So if proving this:  $\Delta PXM \equiv \Delta QYM$ , we can say that the two line segments  $PM$  and  $QM$  have the same lengths; that is, we get this:  $|PM| = |QM|$ , which means that  $\Delta PMQ$  is an isosceles triangle.

Proving thus, this identity:  $\Delta PXM \equiv \Delta QYM$ , and also, this:  $\angle PMQ = 90^\circ$ , we prove that  $\Delta PMQ$  is an isosceles right triangle.

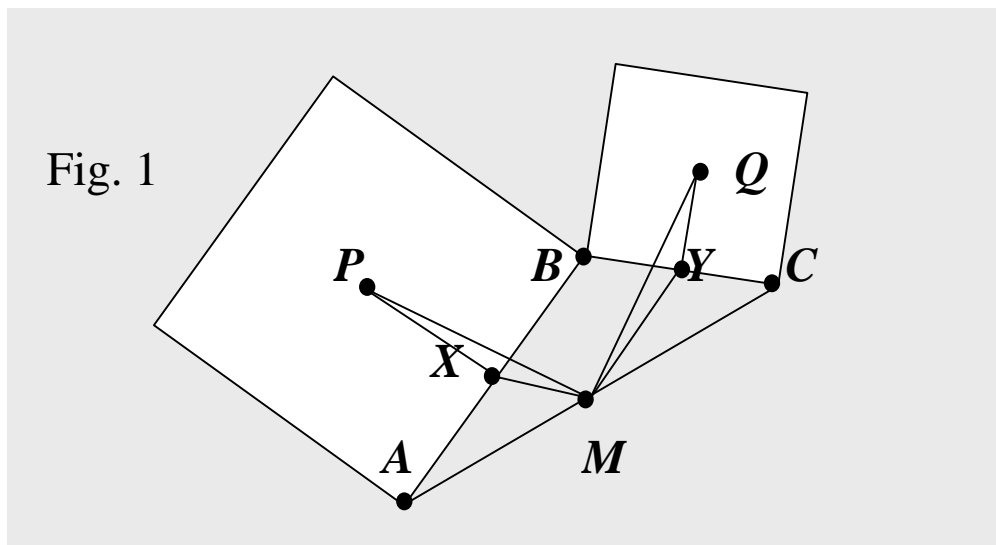
We'll do those two proofs in the next lesson.

## Van Aubel's Theorem Part 3

Let's first, go over the last part in the previous lesson.

To begin with, since the two points  $P$  and  $Q$  are the centers of the squares, we've got these:

$$PX \perp BX, \quad |PX| = |BX|, \quad QY \perp BY, \quad |QY| = |BY|$$



Next, the tetragon  $XMYB$  turned out to be a parallelogram, which means, we've got these, too:

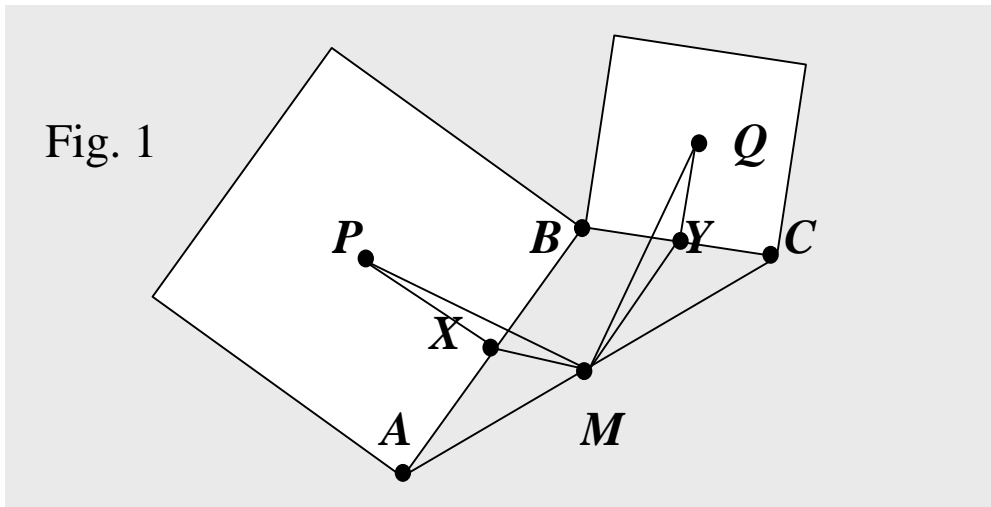
$$|YM| = |BX|, \quad YM \parallel BX, \quad |XM| = |BY|, \quad XM \parallel BY$$

And putting things together simplifying some of those above, we can get these:  $PX \perp BX$ ,  $QY \perp BY$ ,  $YM \parallel BX$ ,  $XM \parallel BY$ ,  $|PX| = |BX| = |YM|$ ,  $|QY| = |BY| = |XM|$

So what are we going to do with the findings shown below?

$$PX \perp BX, \quad QY \perp BY, \quad YM \parallel BX, \quad XM \parallel BY,$$

$$|PX| = |BX| = |YM|, \quad |QY| = |BY| = |XM|$$



As stated in the previous lesson, we are going to prove that  $\triangle PMQ$  is an isosceles right triangle.

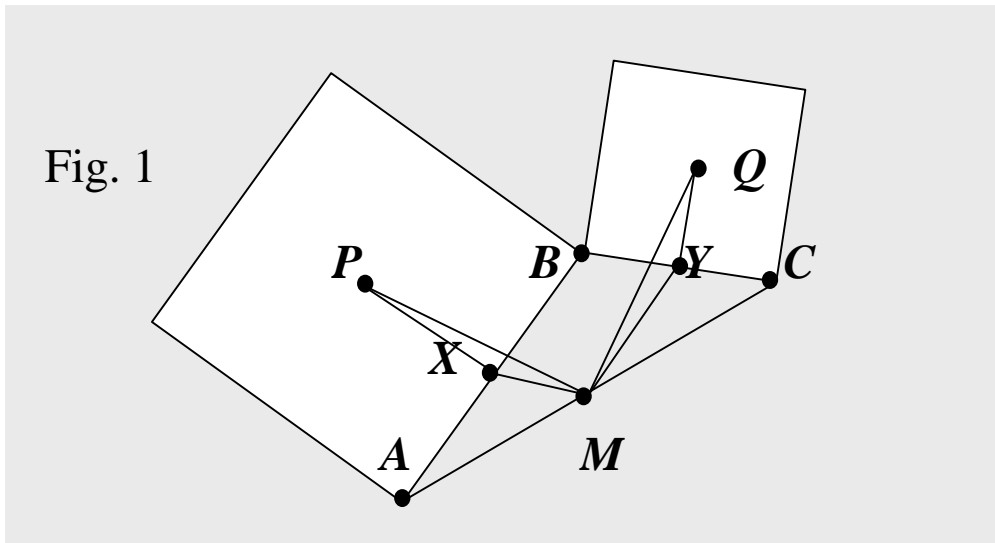
What then do we need to do with the findings above?

If we show this:  $|PM| = |QM|$ , and also, this:  $\angle PMQ = 90^\circ$  using the findings above, we produce the proof.

Let's now begin with this equality:  $|PM| = |QM|$ , which is saying that the two sides have the same lengths.

So what have we got to work with to prove the equality?

Looking at closely the two triangles  $\triangle PXM$  and  $\triangle QYM$ , we can notice that  $PM$  is a side of  $\triangle PXM$  and that  $QM$  is a side of  $\triangle QYM$ .



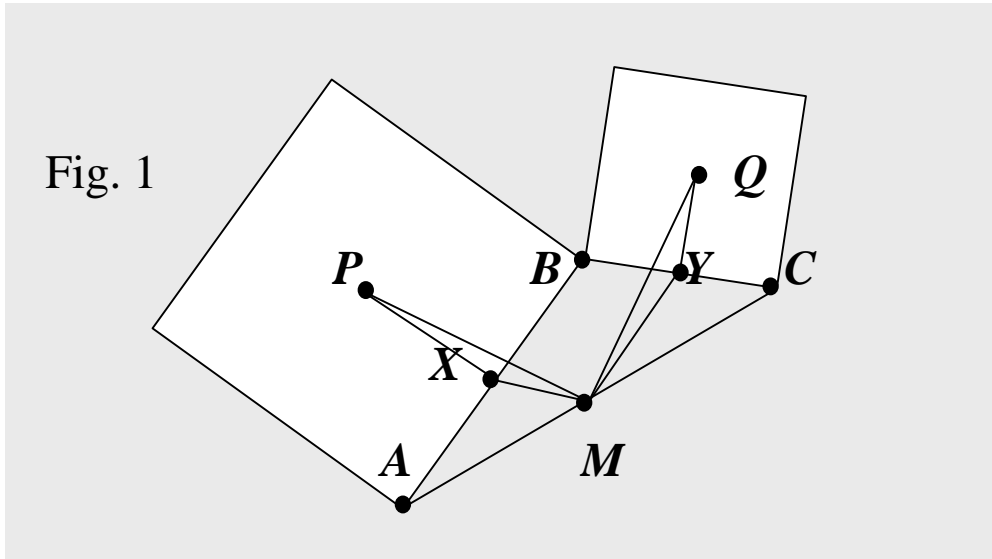
And we have these findings:

$$PX \perp BX, \quad QY \perp BY, \quad YM \parallel BX, \quad XM \parallel BY,$$

$$|PX| = |BX| = |YM|, \quad |QY| = |BY| = |XM|$$

So among those findings above, looking at these two findings:  $|PX| = |BX| = |YM|$  and  $|QY| = |BY| = |XM|$ , we can notice that the two triangles  $\triangle PXM$  and  $\triangle QYM$  share two sides. What then are the two?

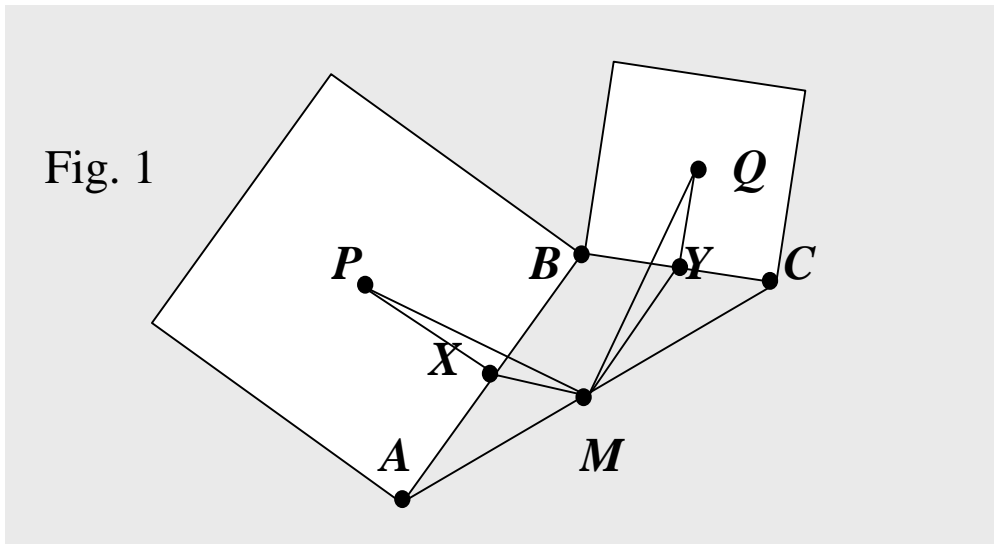
From these:  $|PX| = |BX| = |YM|$  and  $|QY| = |BY| = |XM|$ ,  
 we can get these:  $|PX| = |YM|$  and  $|QY| = |XM|$ .



So we can say that  $\triangle PXM$  and  $\triangle QYM$  share two sides,  
 which are these:  $PX$  and  $XM$ , or these two:  $QY$  and  $YM$ .

So what can we expect?

We can expect that  $\triangle PXM$  and  $\triangle QYM$  are identical; that is, we can expect this:  $\triangle PXM \equiv \triangle QYM$ , which is now thus, a hypothesis to be proven.



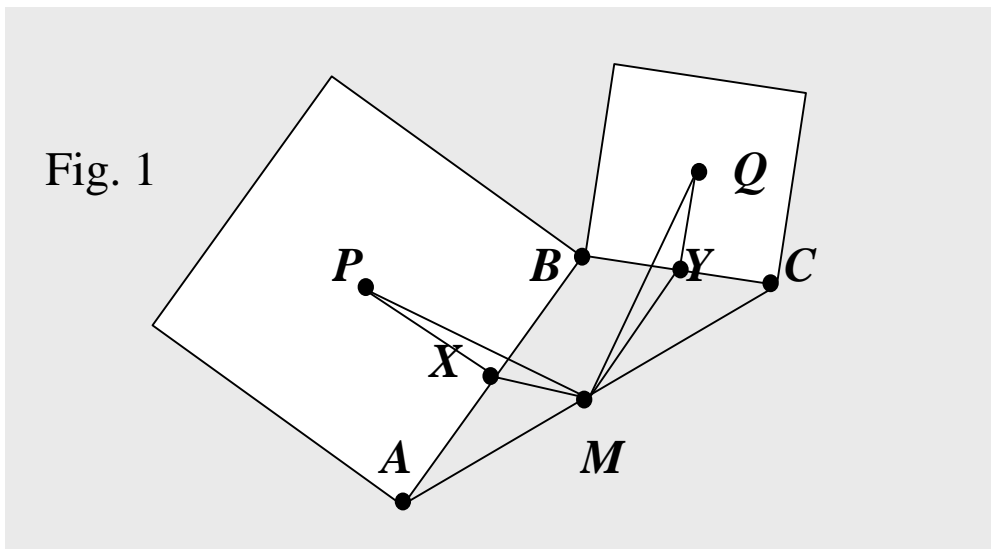
So if proving this:  $\triangle PXM \equiv \triangle QYM$ , we can say that the two line segments  $PM$  and  $QM$  are of the same length; that is, we get this:  $|PM| = |QM|$ , which means that  $\triangle PMQ$  is an isosceles triangle.

Proving thus, this identity:  $\triangle PXM \equiv \triangle QYM$ , and also this:  $\angle PMQ = 90^\circ$ , we prove that  $\triangle PMQ$  is an isosceles right triangle.

How then can we prove the two above?

We do the proof of this identity first:  $\Delta PXM \equiv \Delta QYM$ . And doing it, we can use a basic, which is saying this:

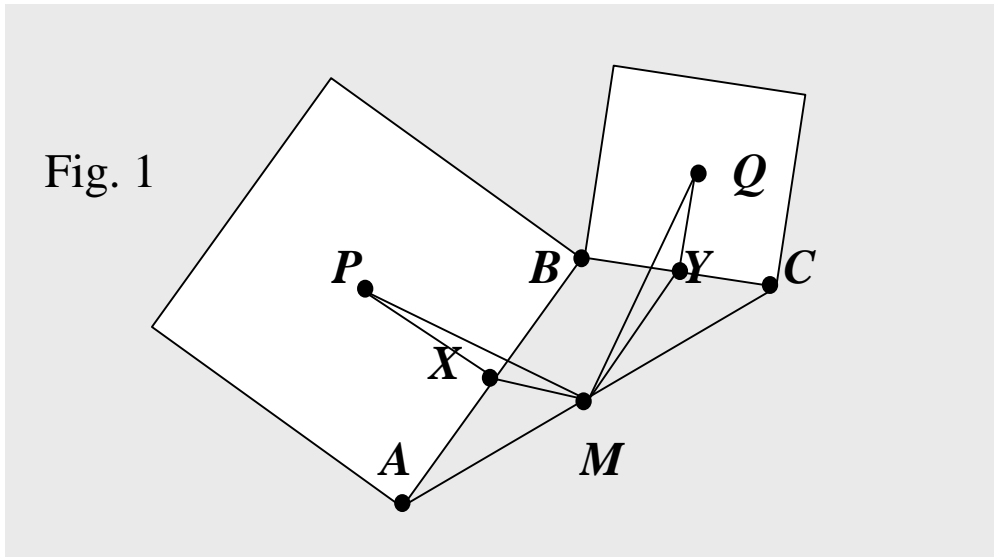
If two triangles share an angle and the two sides making the angle, the two triangles are identical, and vice versa.



So if we prove this:  $\angle PXM = \angle QYM$ , we prove this identity:  $\Delta PXM \equiv \Delta QYM$ , since the triangles share two sides making this angle:  $\angle PXM$ , which equals this:  $\angle QYM$ .

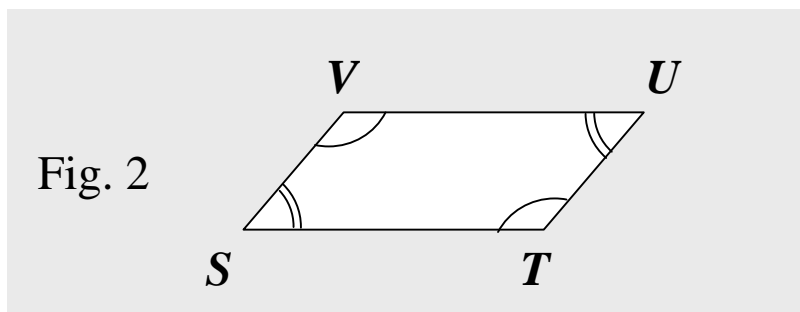
So let's see now, how we can get this:  $\angle PXM = \angle QYM$ .

First, the quadrangle  $XMYB$  is a parallelogram.



And we have another basic as follows.

In a parallelogram, each pair of angles facing each other over its center are the same.

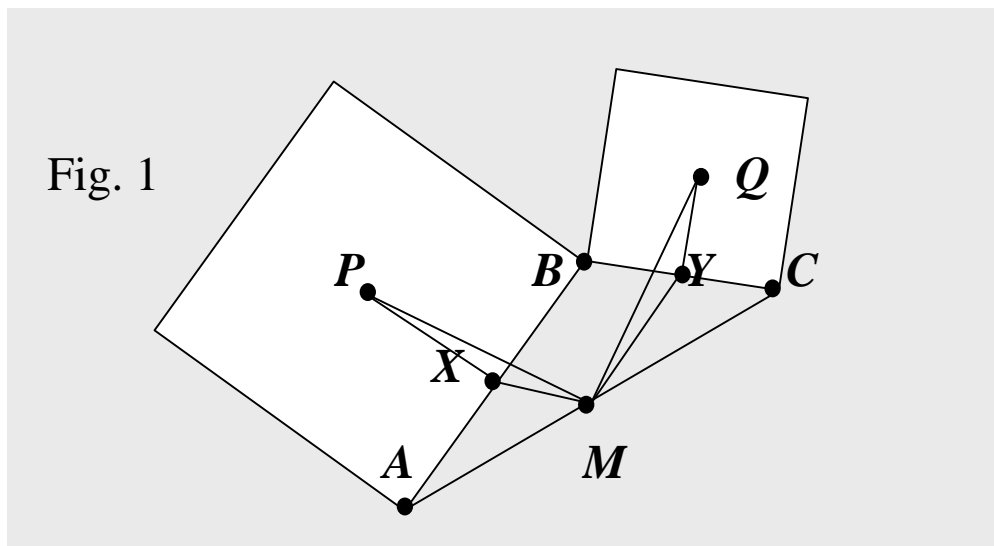


So if the quadrangle  $STUV$  above is a parallelogram, we get these:  $\angle S = \angle U$ , and  $\angle V = \angle T$ .

If not sure of the basic above, that is, if you want to know how it is the case, refer to a section called **Parallelograms** in the lesson notes called **Basic Shapes**.

Now, all set to go, and we can begin with this:

$$\angle PXB = 90^\circ = \angle QYB.$$



So we get these:

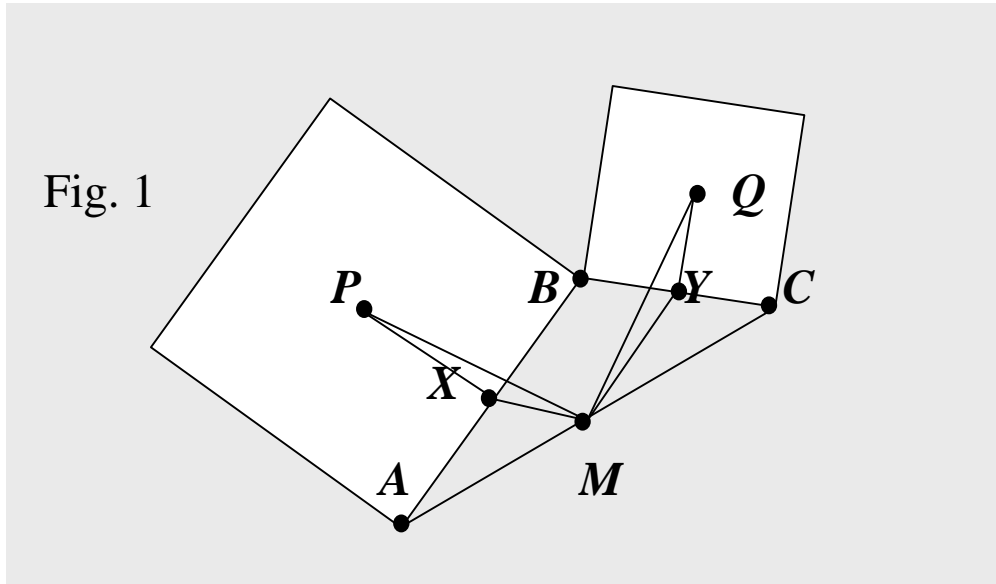
$$\angle PXM = 90^\circ + \angle BXM, \text{ and } \angle QXM = 90^\circ + \angle BYM.$$

And we have this, too:  $\angle BXM = \angle BYM$ . Why, though?

It's because the quadrangle  $XYMB$  is a parallelogram.

So we get this:  $\angle PXM = \angle QYM$ .

Also, we have these:  $|PX| = |YM|$ , and  $|QY| = |XM|$ , along with this:  $\angle PXM = \angle QYM$ .



So we get this:  $\Delta PXM \equiv \Delta QYM$ . Why, though?

As stated above, we have the basic that if two triangles share two sides and the angle between the two sides, the two triangles are identical.

So we can now say that the two triangles  $\Delta PXM$  and  $\Delta QYM$  are exactly the same, identical. That is to say that we get this:  $|PM| = |QM|$ , and thus, the proof is done.

So now, one down one to go.

It's about the angle between  $PM$  and  $QM$ . It is  $90^\circ$ , that is,  $\angle QMP = 90^\circ$ . And of course, we'll show how it is the case. Doing now thus, the proof, we can begin with this:

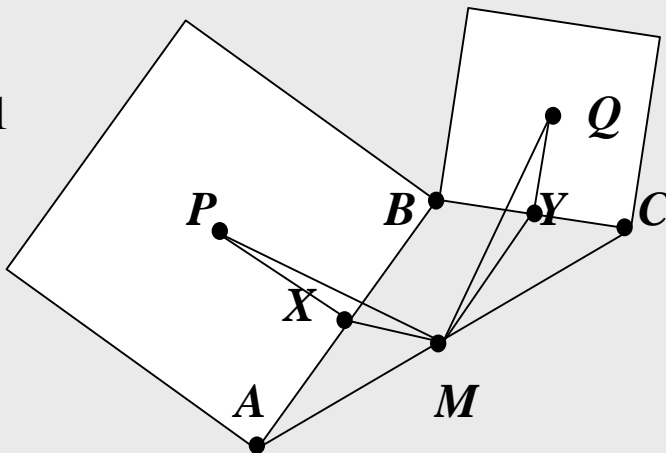
The sum of all the three angles in a triangle is  $180^\circ$ .

The sum of all the four angles in a quadrangle is  $360^\circ$ .

Wonder why beginning with the basics above?

We want to know why it's a right angle,  $90^\circ$ . So it seems we are now working on angles. And looking at the figure below, we can notice that the sides supposed to make the angle of  $90^\circ$  belong to two triangles, and that the triangles are connected with a parallelogram.

Fig. 1



Thus, we may want to check with the basics on angles in a triangle and a quadrangle called a parallelogram. And some of the basics on the angles in a triangle are as follows.

The sum of all the three angles in a triangle is  $180^\circ$ .

The sum of all the four angles in a quadrangle is  $360^\circ$ .

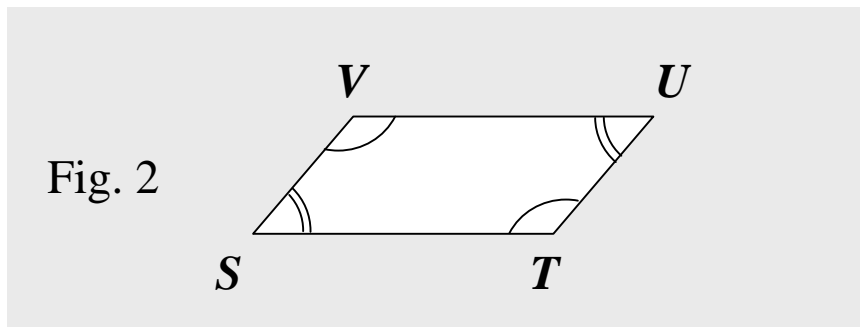
If not sure of how it is the case, that is, if you want to know how the basics are true, you may want to refer to a section called ***Triangles*** in the lesson notes called ***Basic Shapes***.

What then about the basics on angles in a parallelogram?

We have another math basic, which is saying that in a parallelogram, every pair of angles next to each other add up to  $180^\circ$ , which can be said to be a condition for a parallelogram, and also, one of its properties.

So for instance, in the figure below, if the quadrangle *STUV* is a parallelogram, we get this:

$$\angle S + \angle T = \angle T + \angle U = \angle U + \angle V = \angle V + \angle S = 180^\circ.$$

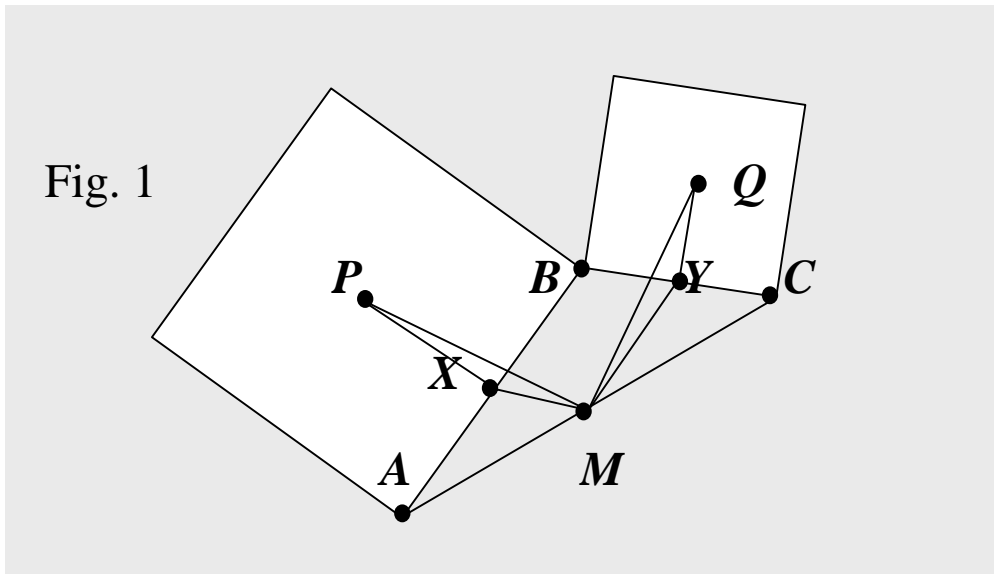


And as shown in the figure above, in a parallelogram, each pair of angles facing each other over its center are the same, which is another math basic often used.

If not quite sure of how the two basics are true, you may want to refer to a section called **Parallelograms** in the lesson notes called **Basic Shapes**.

So now, backing to Fig. 1 shown below and using the basic that says this: in a parallelogram, every pair of angles next to each other add up to  $180^\circ$ , we get this:

$$(\angle YMQ + \angle QMP + \angle XMP) + \angle BXM = 180^\circ \dots (1)$$



Next, we have another basic that says this: the sum of the three angles in a triangle is  $180^\circ$ , so we get this:

$$\angle XMP + \angle XPM + (\angle BXM + \angle PXB) = 180^\circ.$$

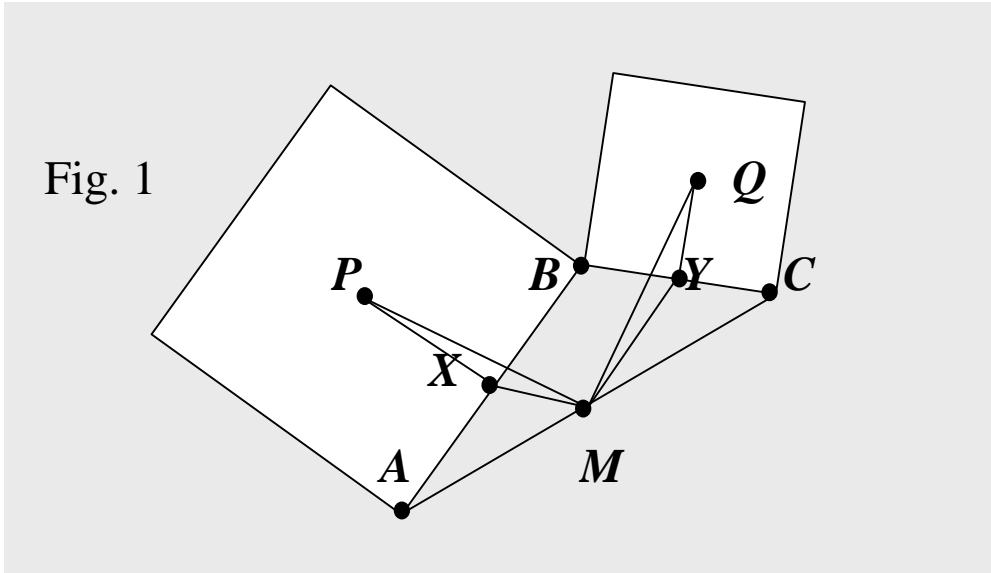
And we have this:  $\angle PXB = 90^\circ$ .

So we get this:  $\angle XMP + \angle XPM + (\angle BXM + 90^\circ) = 180^\circ$ .

Thus, we get this:  $\angle XMP + \angle XPM + \angle BXM = 90^\circ \dots (2)$

Now, getting back to (1), we have this:

$$\angle YMQ + \angle QMP + \angle XMP + \angle BXM = 180^\circ \dots (1)$$



And we have this, too:  $\angle YMQ = \angle XPM$ .

So from (1), we get this:

$$\angle XPM + \angle QMP + \angle XMP + \angle BXM = 180^\circ \dots (3)$$

We now have two interim results as follows.

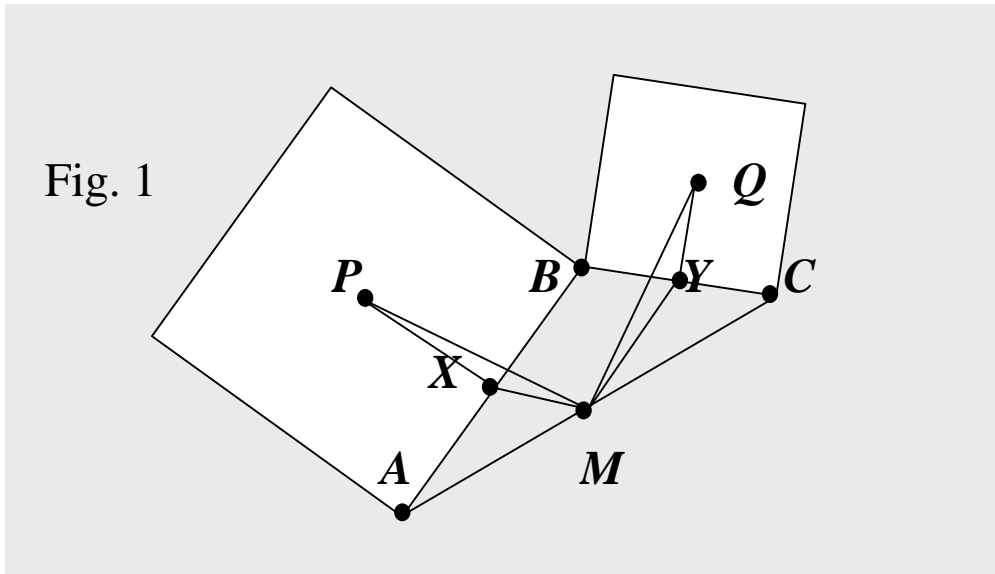
$$\angle XMP + \angle XPM + \angle BXM = 90^\circ \dots (2)$$

$$\angle XPM + \angle QMP + \angle XMP + \angle BXM = 180^\circ \dots (3)$$

So, for now, we have these:

$$\angle XMP + \angle XPM + \angle BXM = 90^\circ \dots (2)$$

$$\angle XPM + \angle QMP + \angle XMP + \angle BXM = 180^\circ \dots (3)$$



Next, rearranging the terms in (3), we can put it this way:

$$\angle XMP + \angle XPM + \angle BXM + \angle QMP = 180^\circ \dots (4)$$

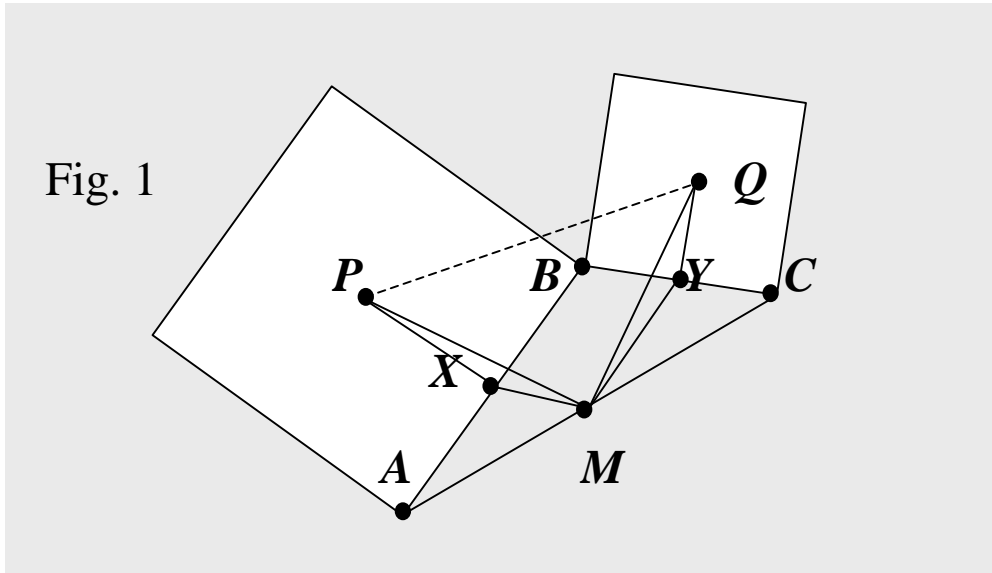
Then, using (2), that is,  $\angle XMP + \angle XPM + \angle BXM = 90^\circ$ ,

we can put (4) this way:

$$(\angle XMP + \angle XPM + \angle BXM) + \angle QMP = 180^\circ.$$

$\Rightarrow 90^\circ + \angle QMP = 180^\circ$ , that is, we get this:  $\angle QMP = 90^\circ$ .

So  $\triangle QMP$  is an isosceles right triangle.

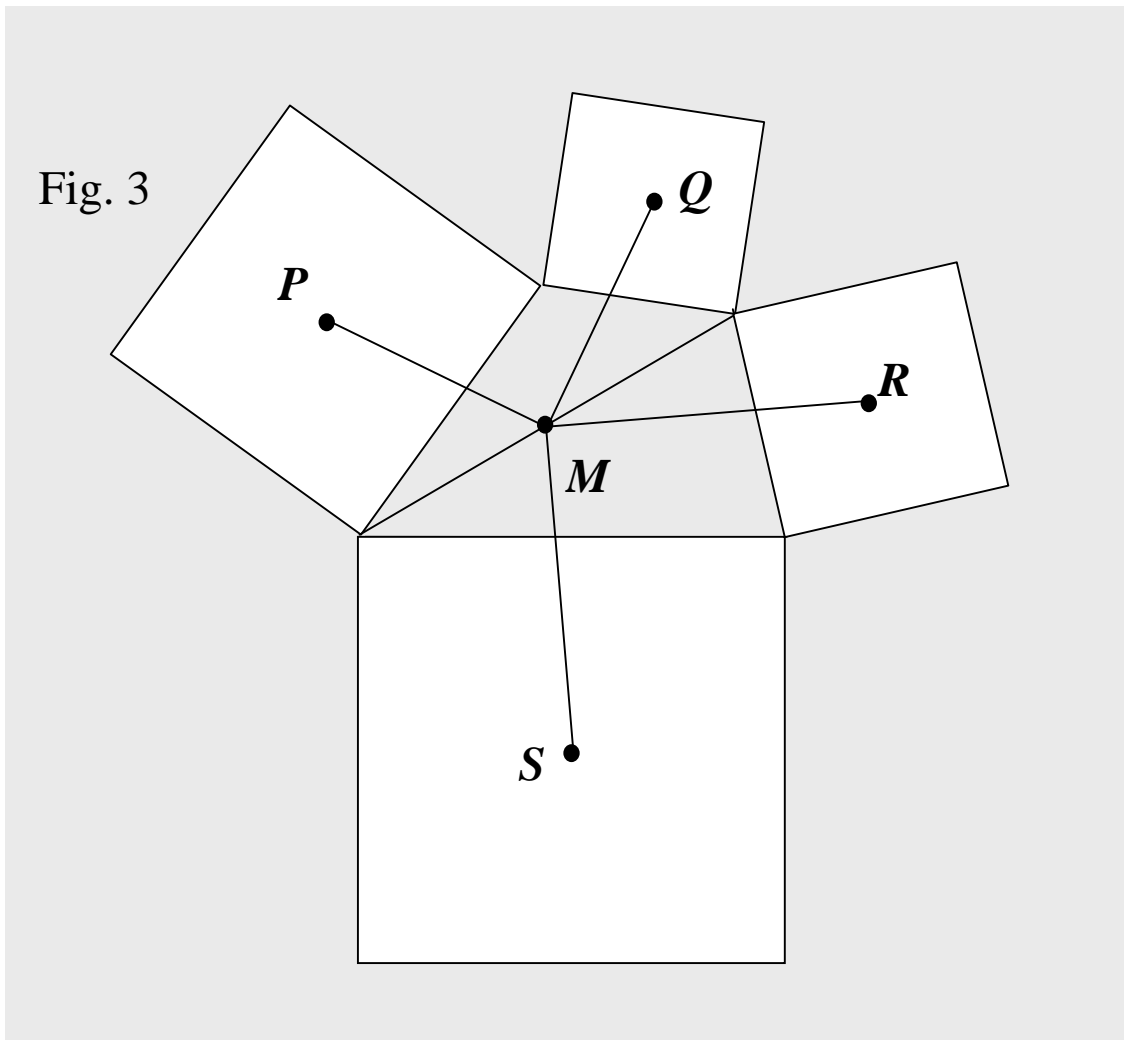


Besides, the two triangles  $\triangle PXM$  and  $\triangle QYM$  are identical.

So we can now say that putting together two squares and a triangle the way above and connecting the centers of the squares and the midpoint of the other side of the triangle, we get an **isosceles right triangle**.

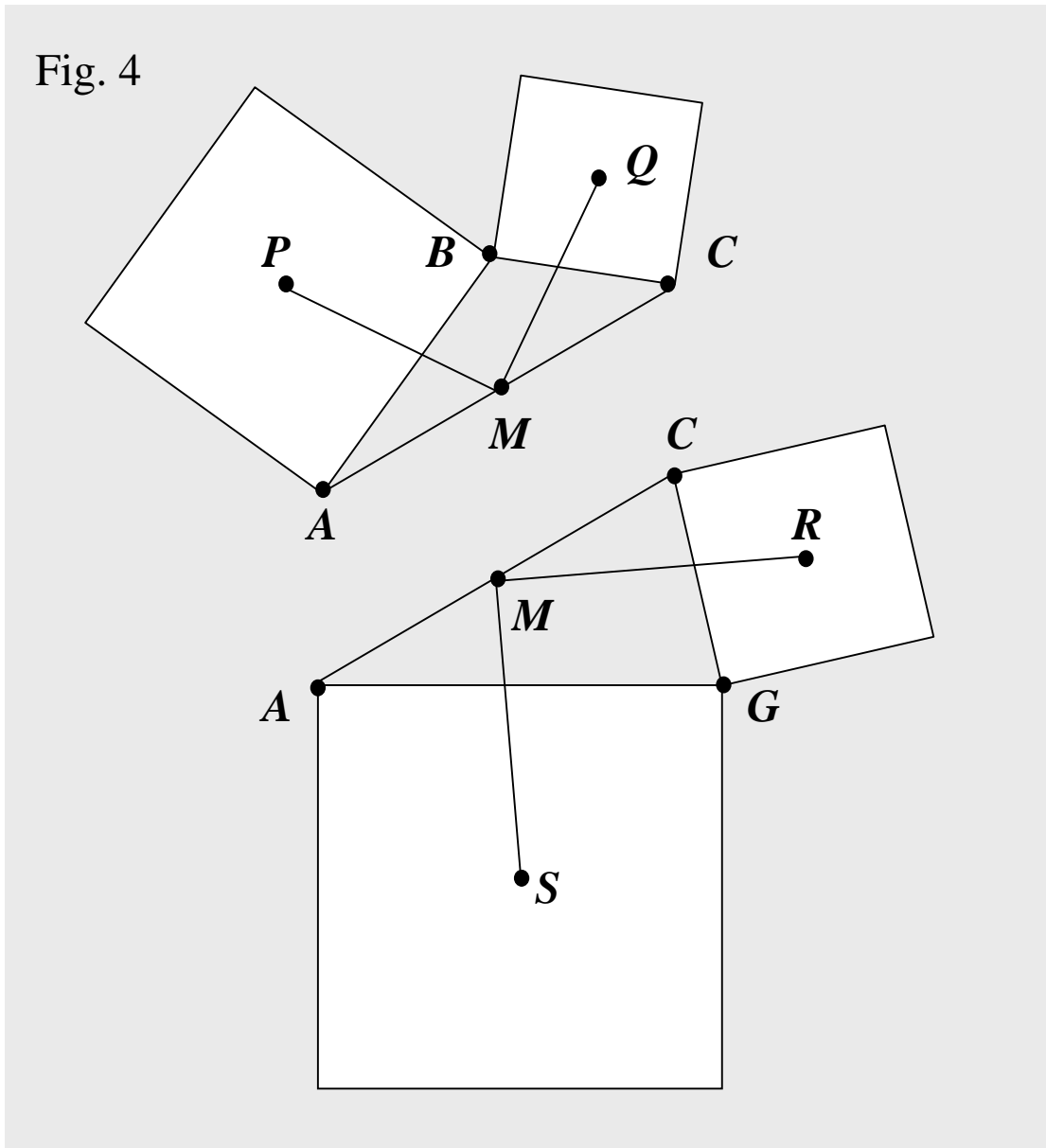
Also, we get two identical triangles each of which takes as a side each of the two legs of the right triangle isosceles.

Let's now take a step further, and take a look at the next situation, where we put together the two sets, each of which is made of two squares and a triangle. So now, putting the two sets together, we get this:



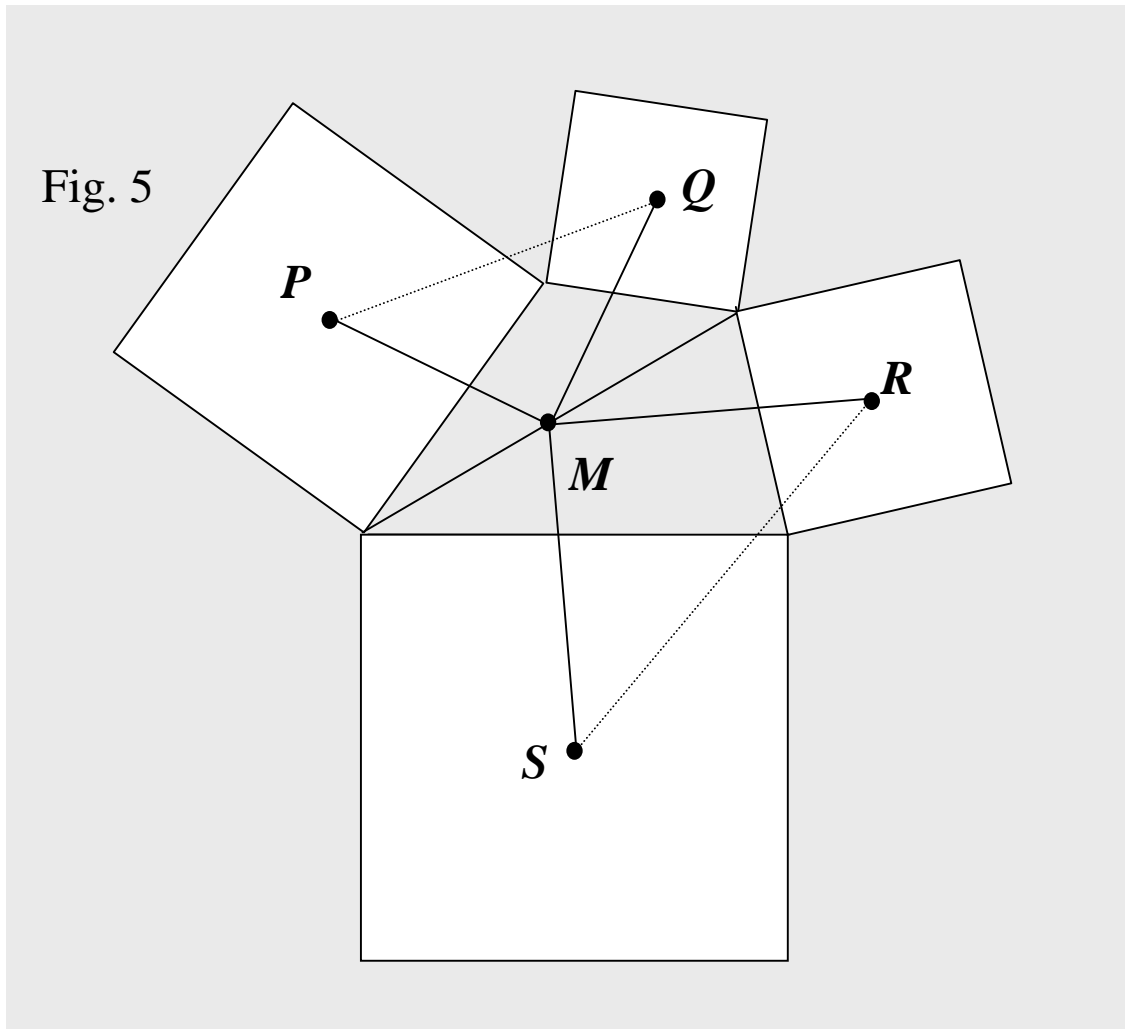
In the figure above,  $R$  and  $S$  are the two centers of the two squares in the lower set.

If separating one set from the other, we can put the two the way as follows.



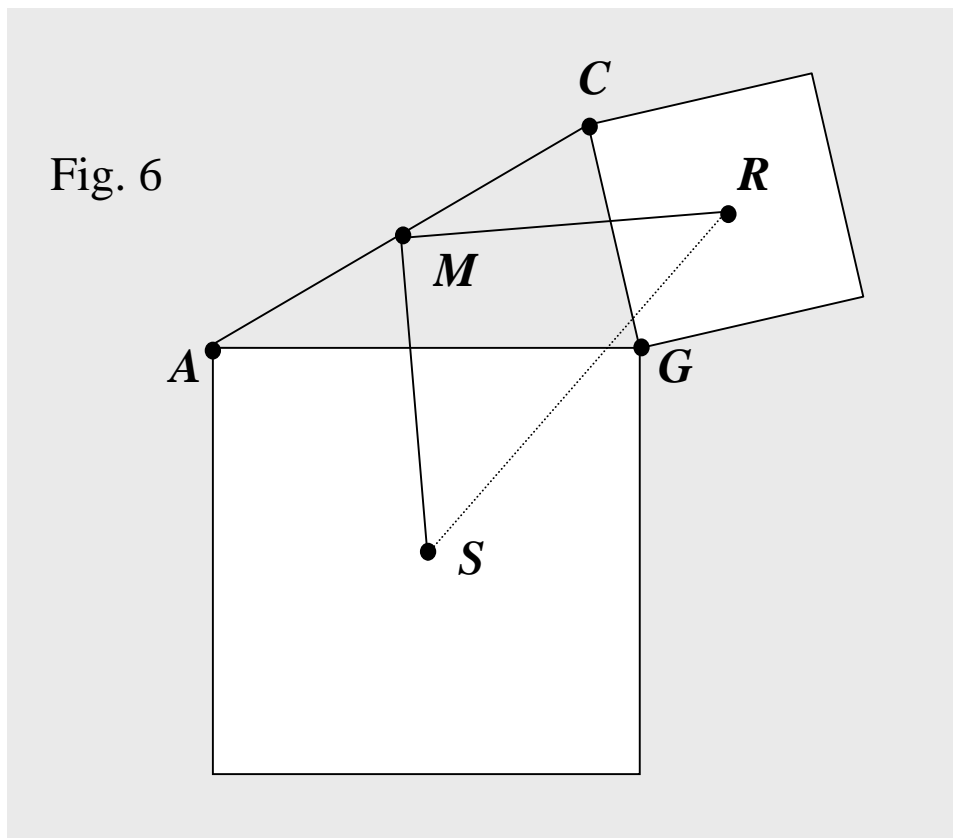
The method used for the upper set can also be used for the lower set.

Putting the two sets together, we get the figure as follows.



So, the point is this: We can apply to the lower set, too, the method used for the upper. Using, therefore, the same method we used to show that  $\Delta PMQ$  is an isosceles right triangle where  $\angle PMQ = 90^\circ$ , we can show that  $\Delta SMR$  is an isosceles right triangle, too, where  $\angle SMR = 90^\circ$ .

It's because the set with the centers  $S$  and  $R$  is no other than the set with the centers  $P$  and  $Q$ . The two sets are practically the same, so we can move on to the next step without proving that  $\Delta SMR$  is an isosceles right triangle, too.

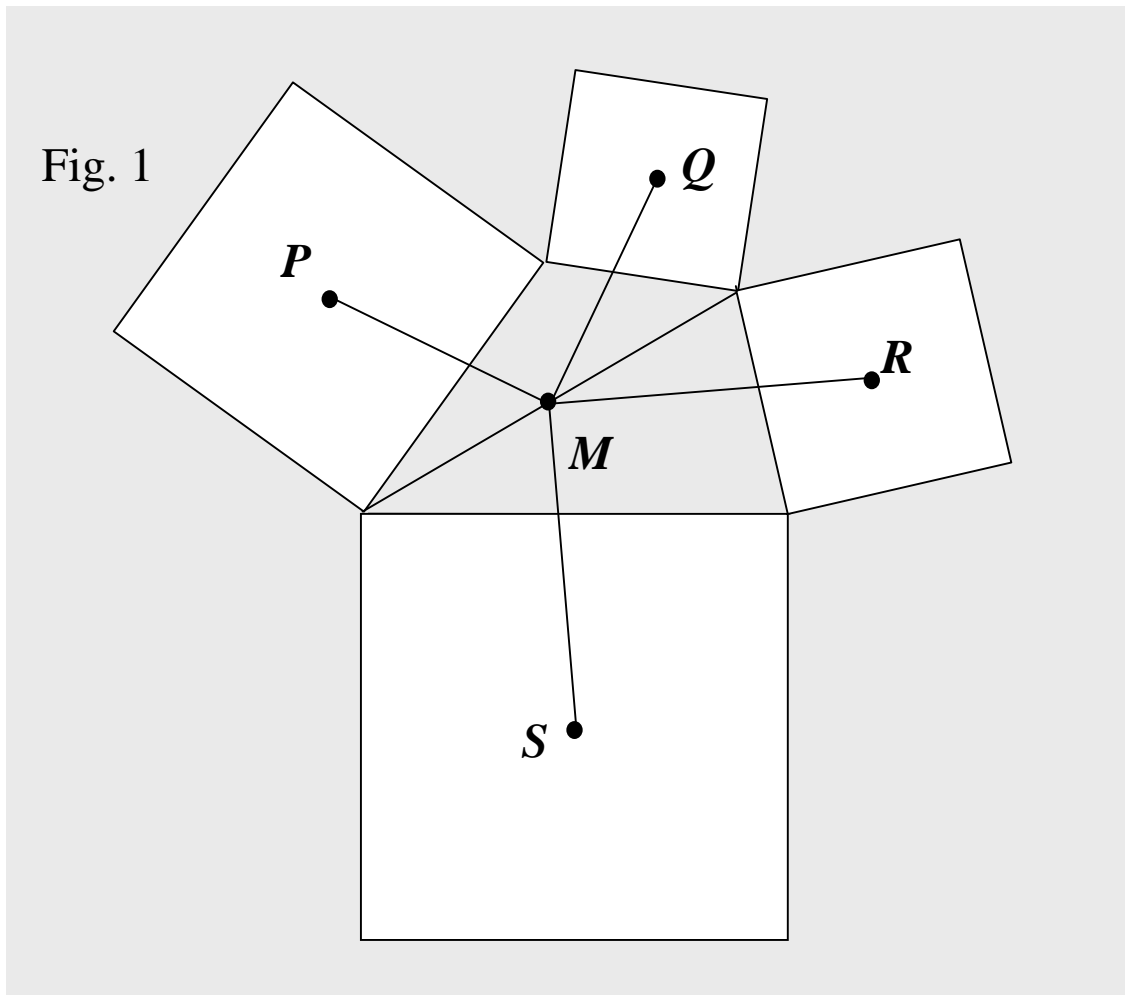


We'll see in the next lesson, though, as a practice, how the same method can be applied to the lower set, that is, we are going to repeat the same proving processes to prove that  $\Delta SMR$  is an isosceles right triangle where  $\angle SMR = 90^\circ$ .

## Van Aubel's Theorem Part 4

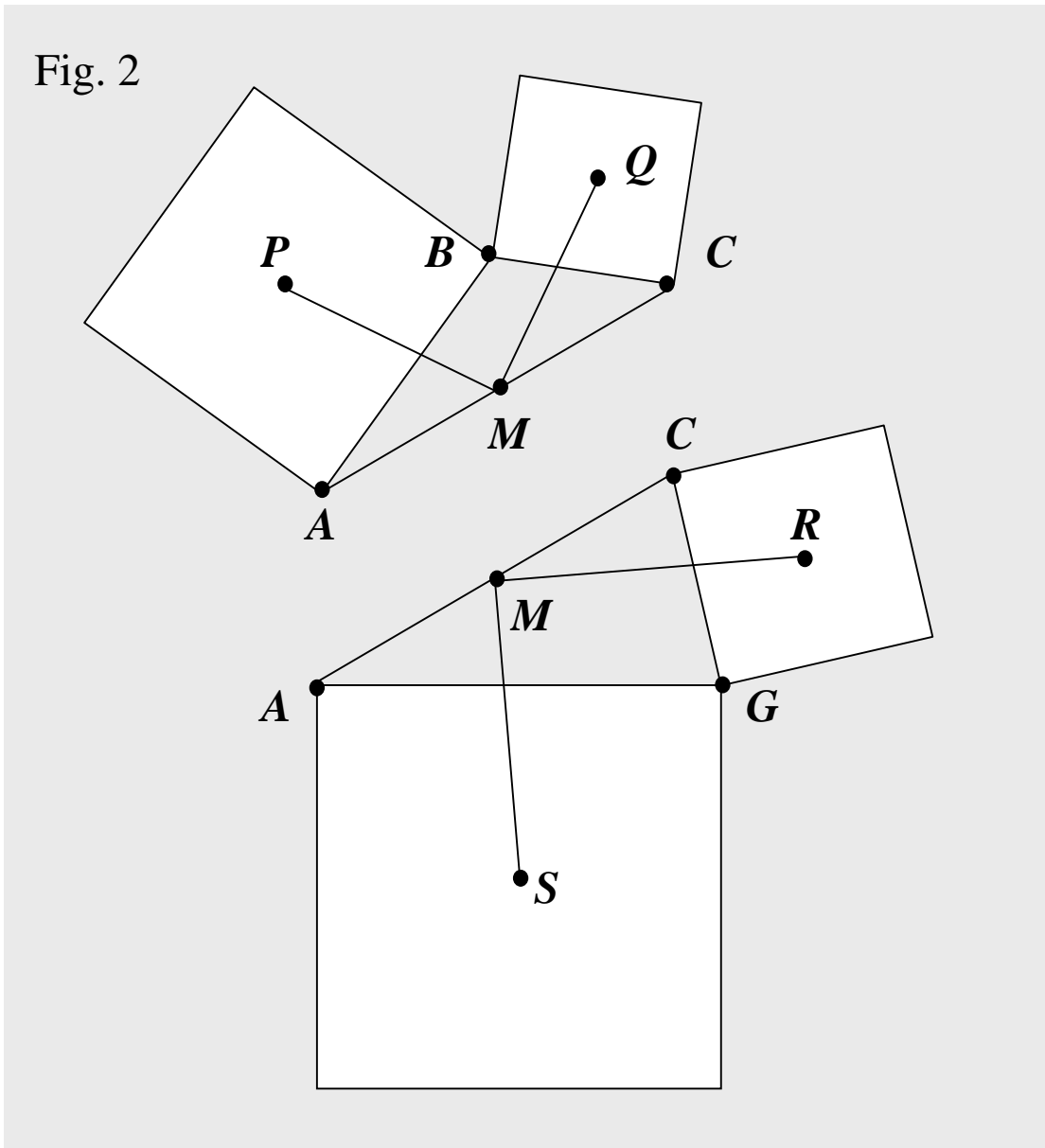
Let's first, go over the last part in the previous lesson.

Putting together the two sets each of which is made of two squares and a triangle, we can get the figure below.



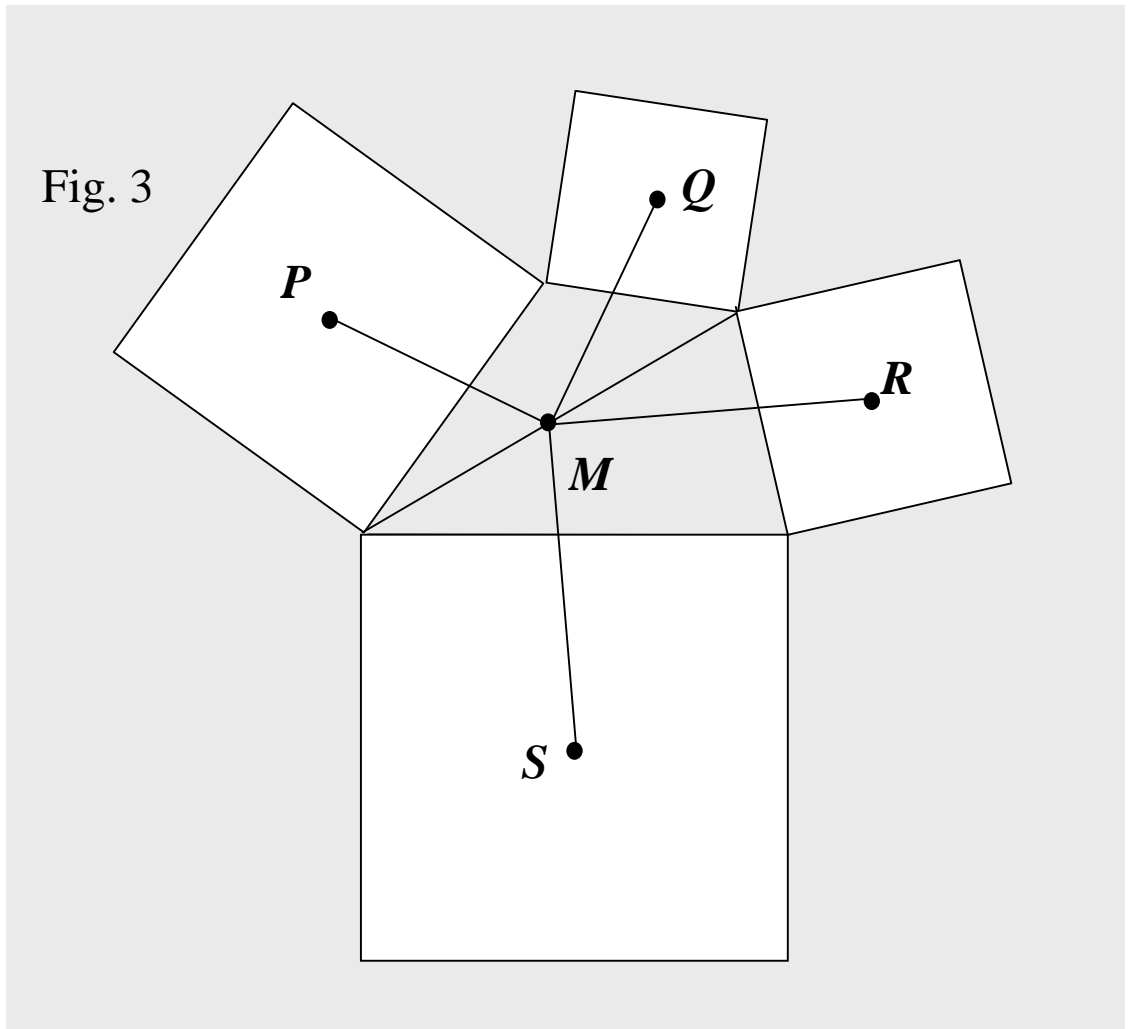
In the figure above,  $R$  and  $S$  are the two centers of the two squares in the lower set.

If separating one set from the other, we can put the two the way as follows.



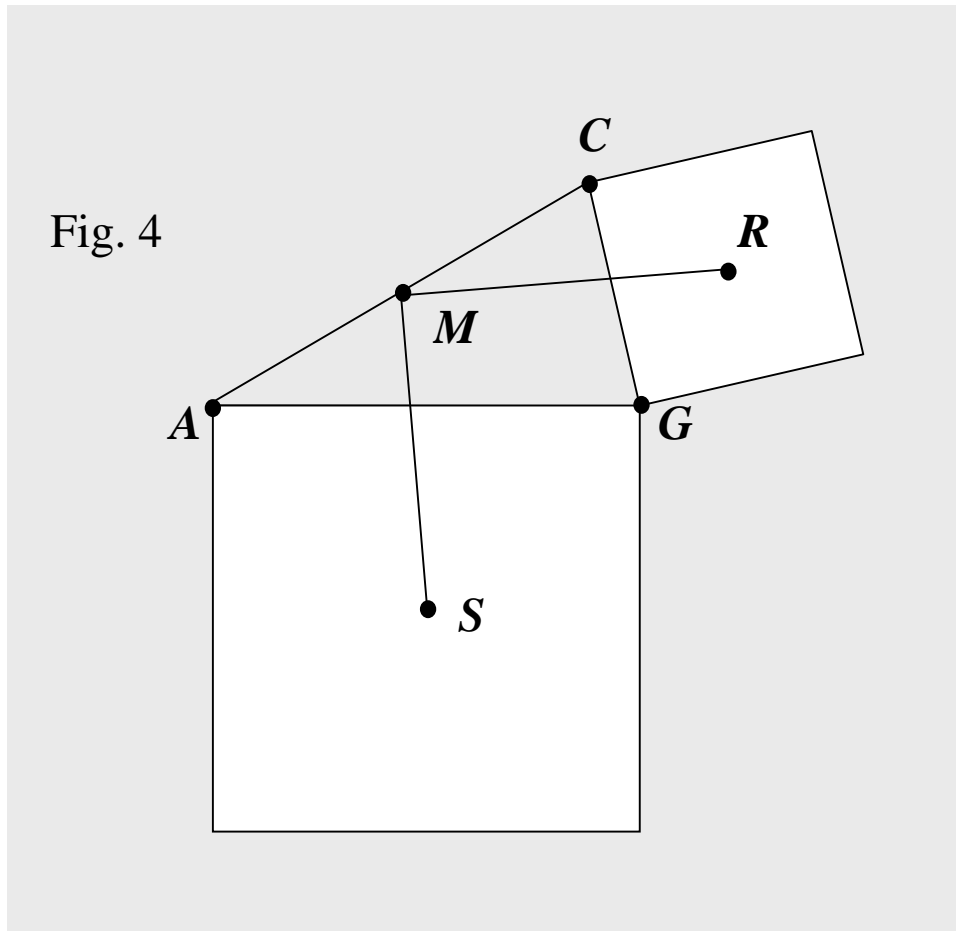
The method used for the upper set can also be used for the lower set.

Putting the two sets together, we get the figure as follows.



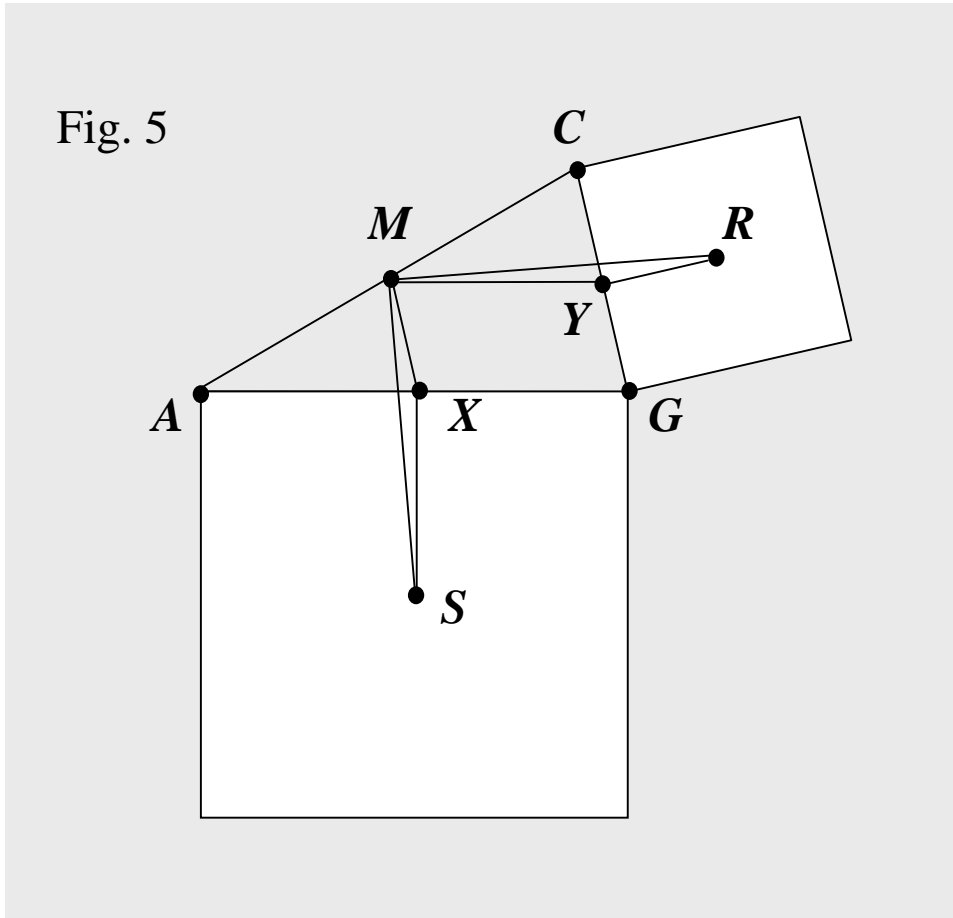
We can apply to the lower set the method used for the upper. So using the same method we used to show that  $\Delta PMQ$  is an isosceles right triangle where  $\angle PMQ = 90^\circ$ , we can show that  $\Delta SMR$  is an isosceles right triangle, too, where  $\angle SMR = 90^\circ$ .

It's because the set with the centers  $S$  and  $R$  is no other than the set with the centers  $P$  and  $Q$ .



Let's now take a look at how the same method can be applied to the lower set, too. So in the same method, we'll get these two:  $|SM| = |RM|$ , and  $\angle SMR = 90^\circ$ . And we are going to start approaching this first:  $|SM| = |RM|$ .

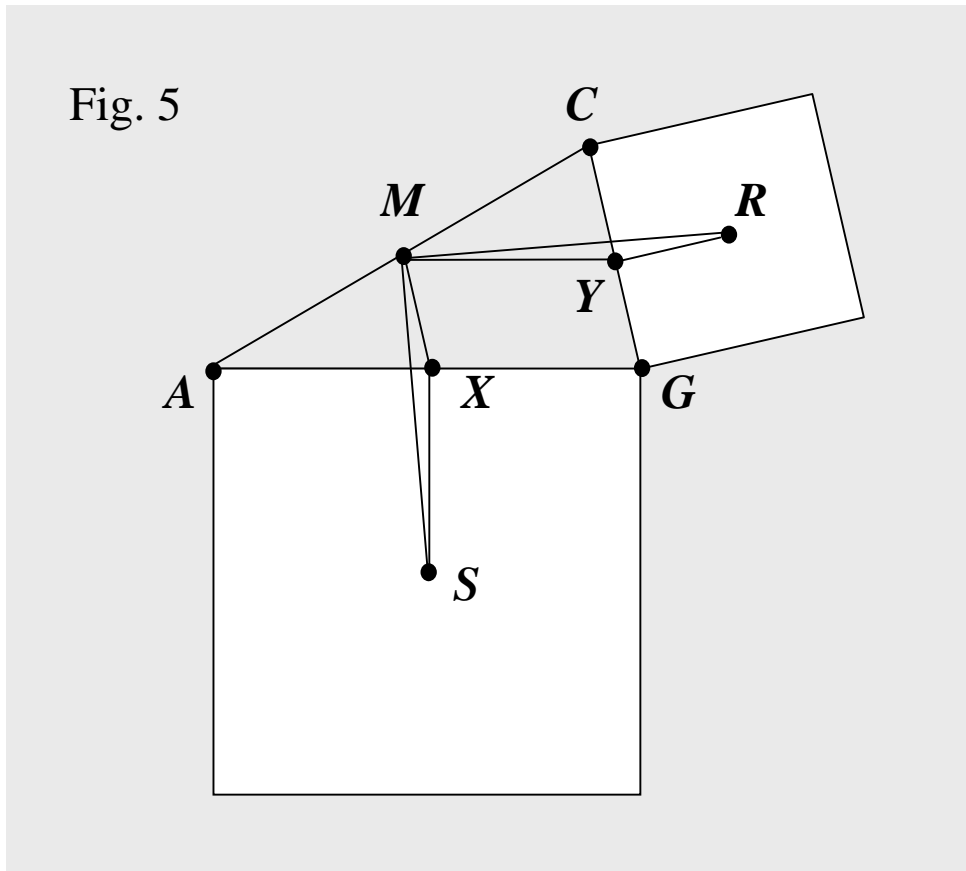
Assuming now,  $X$  is the midpoint of  $AG$ , and  $Y$  is the midpoint of  $GC$ , and connecting some points with line segments, we can get the figure below.



Then, since  $X$  is the midpoint of  $AG$ , and  $Y$  is the midpoint of  $GC$ , and  $S$  and  $R$  are the centers of the squares, we get this:  $SX \perp XG$ ,  $|SX| = |XG|$ ,  $RY \perp YG$ , and  $|RY| = |YG|$ .

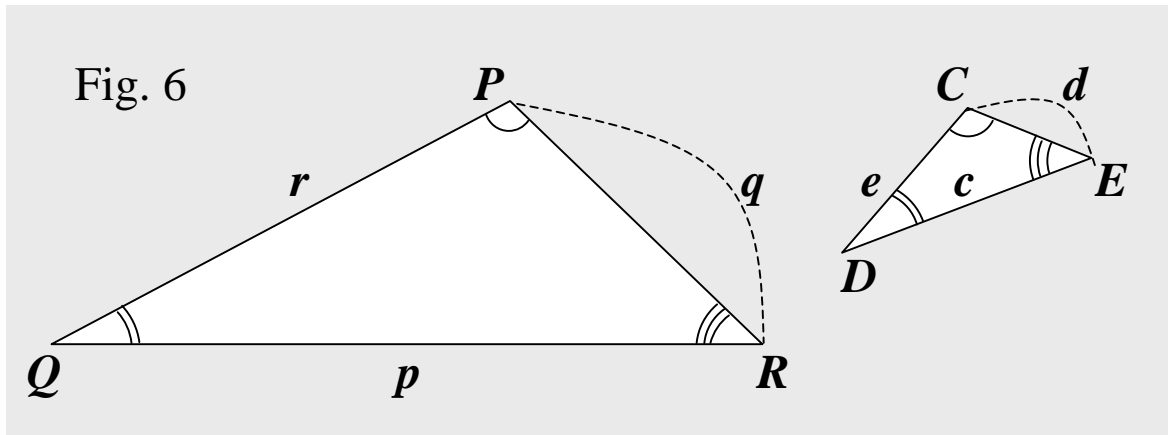
That is to say that the two line segments  $SX$  and  $GX$  meet at a right angle and are of the same length, and the same is true of the two line segments  $RY$  and  $GY$ , too.

What then about  $MY$  and  $MX$ ?



We get this:  $|MY| = |XG|$ , and  $|MX| = |YG|$ . How?

We have math basics called **definitions for similar triangles**, which are in two groups. Each group is made of three versions, and one group is as follows.



$$(1) \quad \Delta PQR \sim \Delta CDE \Leftrightarrow \frac{c}{p} = \frac{d}{q} = \frac{e}{r}$$

$$(2) \quad \Delta PQR \sim \Delta CDE \Leftrightarrow \angle P = \angle C, \text{ and } \angle Q = \angle D$$

$$(3) \quad \Delta PQR \sim \Delta CDE \Leftrightarrow \angle P = \angle C, \text{ and } \frac{e}{r} = \frac{d}{q}$$

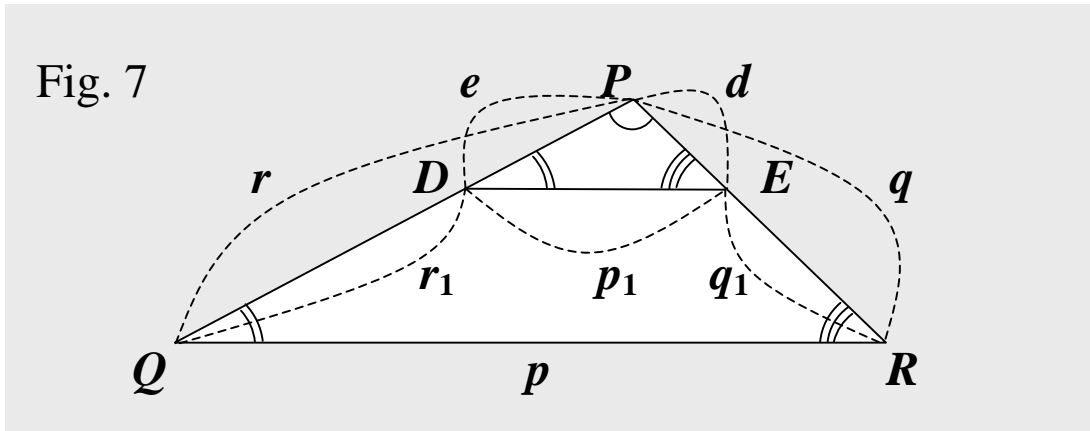
And if any of the three above is true, we get these:

$$(a) \quad \frac{c}{p} = \frac{d}{q} = \frac{e}{r}, \text{ three equal ratios, the same scale factor}$$

$$(b) \quad \angle P = \angle C, \angle Q = \angle D, \text{ and } \angle R = \angle E,$$

i.e., both triangles have the same angle groups.

And the other group is as follows. Note that  $\triangle PDE$  is nested inside  $\triangle PQR$ , and **very particularly** that  $\angle P$  is common.



$$(1-1) \quad \triangle PQR \sim \triangle PDE \quad \Leftrightarrow \quad \frac{d}{q} = \frac{e}{r}$$

$$(2-1) \quad \triangle PQR \sim \triangle PDE \quad \Leftrightarrow \quad \angle Q = \angle D$$

$$(3-1) \quad \triangle PQR \sim \triangle PDE \quad \Leftrightarrow \quad \frac{e}{r_1} = \frac{d}{q_1}$$

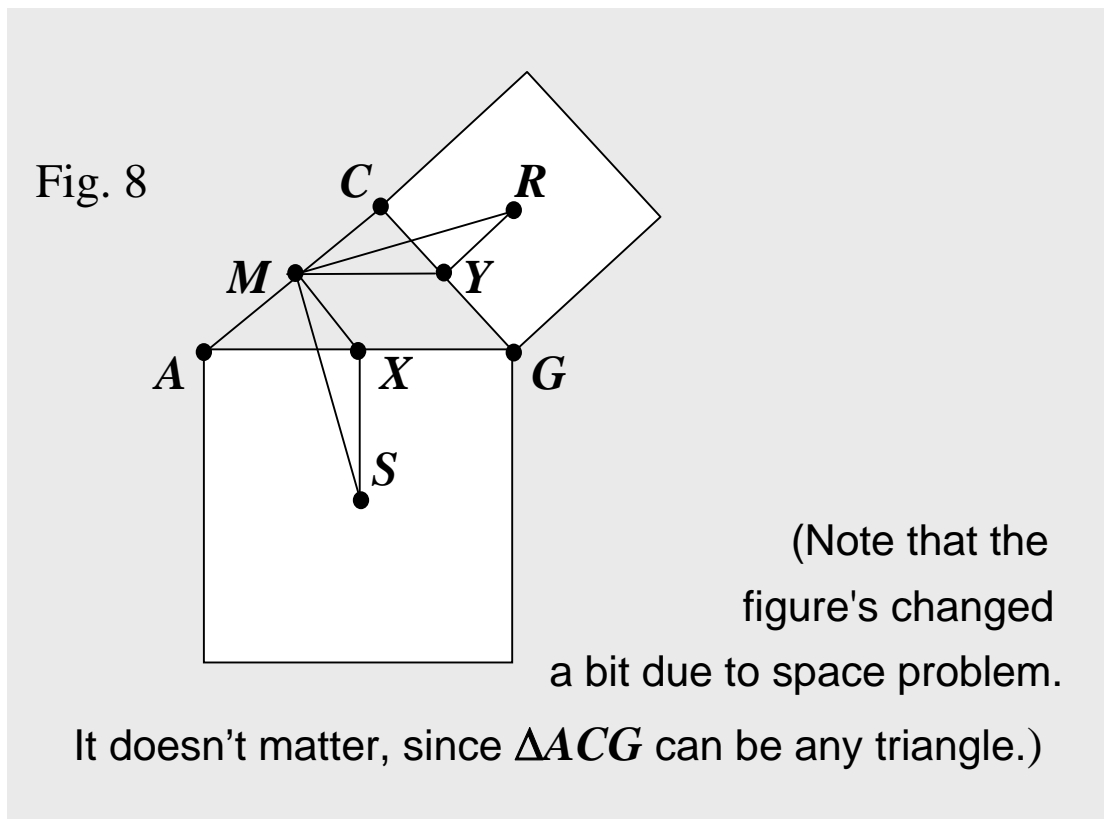
And if any of the three above is true, we get these:

$$(a-1) \quad \frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}, \text{ three equal ratios, the same scale factor}$$

$$(b-1) \quad \angle Q = \angle D, \text{ and } \angle R = \angle E,$$

i.e., both triangles have the same angle groups.

So now, since the three points  $X$ ,  $Y$ , and  $M$  are midpoints of the three sides of  $\triangle ACG$ , we can expect that  $\triangle MCY$  and  $\triangle ACG$  are similar, and we can check if they are similar using one of the definitions above. Which one, though?



We know that  $\triangle MCY$  is nested inside  $\triangle ACG$ , and we have enough information for the ratios between the sides, so we can use the definition (1-1) shown below.

$$(1-1) \quad \triangle PQR \sim \triangle PDE \iff \frac{d}{q} = \frac{e}{r}$$

And if it's true, we can use (a-1) below to see what  $MY$  is.

Note that  $\angle C$  is common to  $\triangle MCY$  and  $\triangle ACG$ , and that  $\triangle MCY$  matches  $\triangle PDE$ , and  $\triangle ACG$  matches  $\triangle PQR$ .

Fig. 8

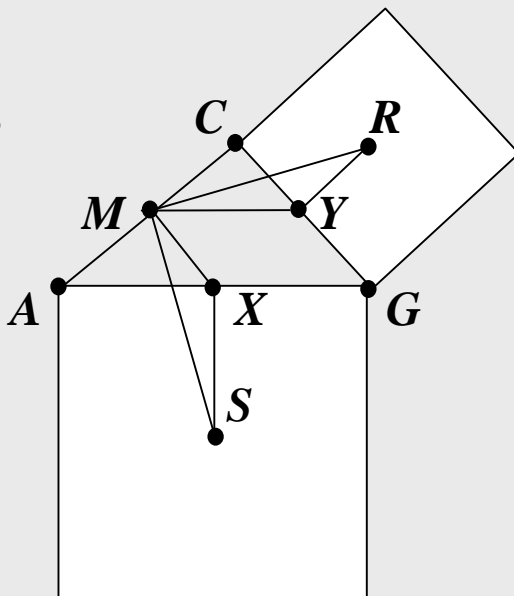
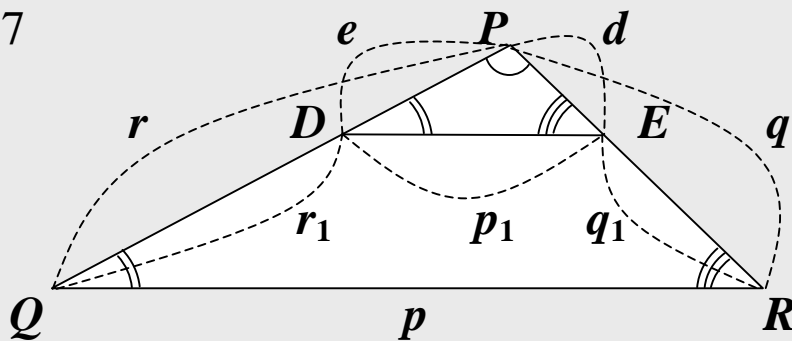


Fig. 7

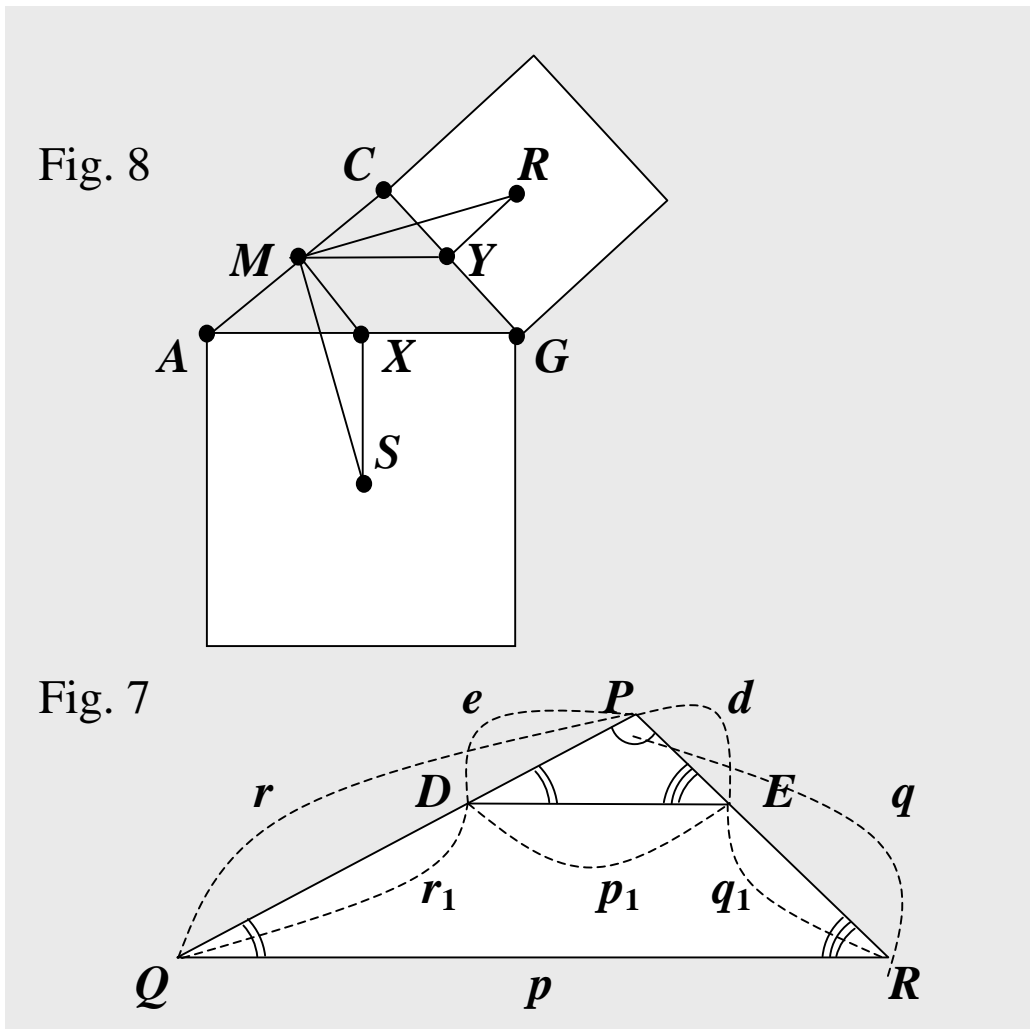


$$(a-1) \quad \frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}, \text{ three equal ratios, the same scale factor}$$

Checking thus, to see if  $\triangle MCY \sim \triangle ACG$  using (1-1) below,

we get this:  $\frac{|CY|}{|CG|} = \frac{1}{2} = \frac{|CM|}{|CA|}$ , so they are similar.

Note that in Fig. 8,  $CY$  matches  $d$  in Fig 7,  $CG$  matches  $q$ ,  $CM$  matches  $e$ , and  $CA$  matches  $r$ .

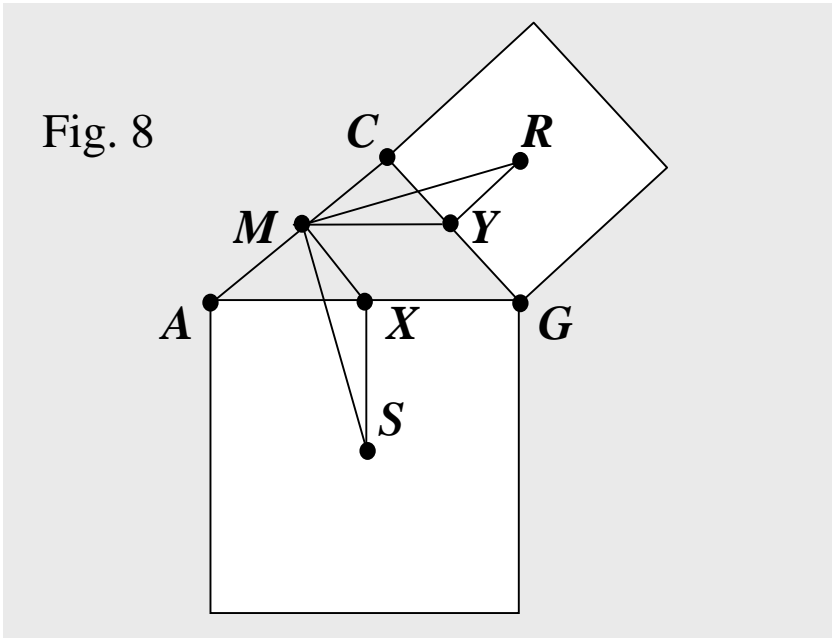


$$(1-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \frac{d}{q} = \frac{e}{r}$$

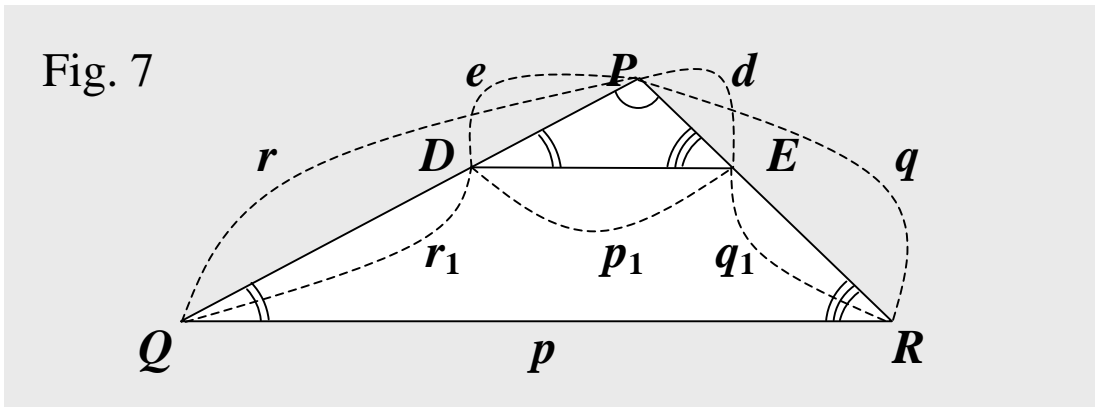
So using (a-1) where  $p_1$  matches  $MY$ , and  $p$  matches  $AG$ ,

we get this:  $\frac{|MY|}{|AD|} = \frac{1}{2}$ . That is, we get this:  $|MY| = \frac{|AG|}{2}$ ,

which is  $|XG|$ . So we get this:  $|MY| = |XG|$ .



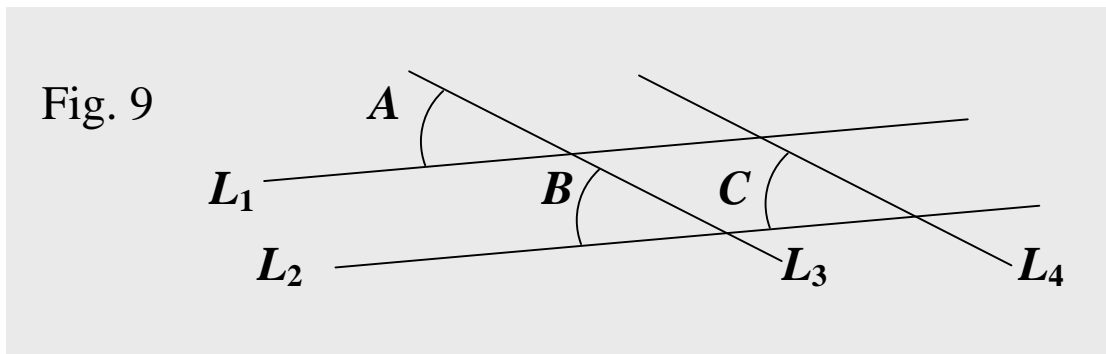
(a-1)  $\frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}$ , three equal ratios, the same scale factor



And, of course, we get this, too:  $MY \parallel XG$ . Why parallel?

A basic says, "If corresponding angles are equal, the lines crossed by the transversal are parallel, and vice versa."

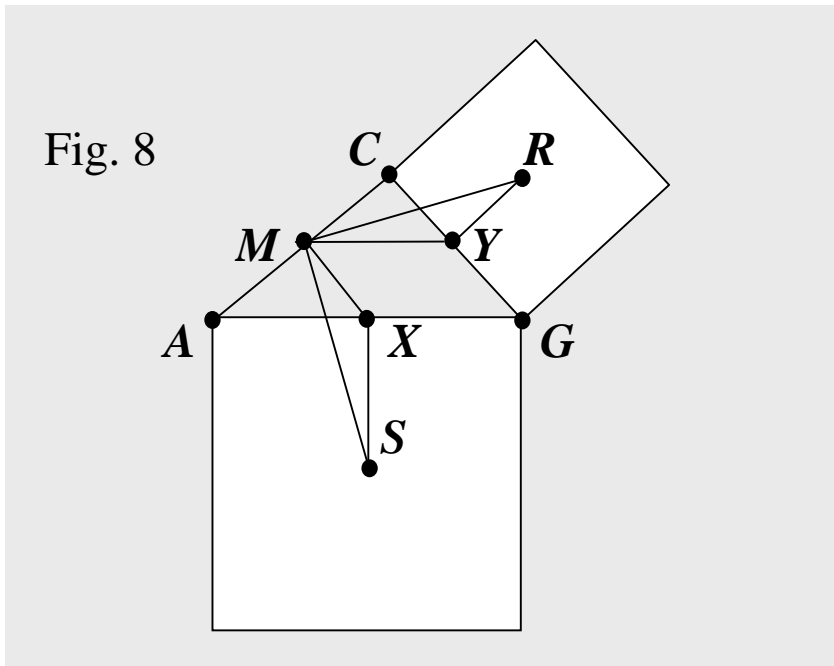
So in Fig. 9 below, if we have this:  $\angle A = \angle B$ , we get this:  $L_1 \parallel L_2$ , because  $\angle A$  and  $\angle B$  are corresponding angles, since  $L_3$  is the transversal.



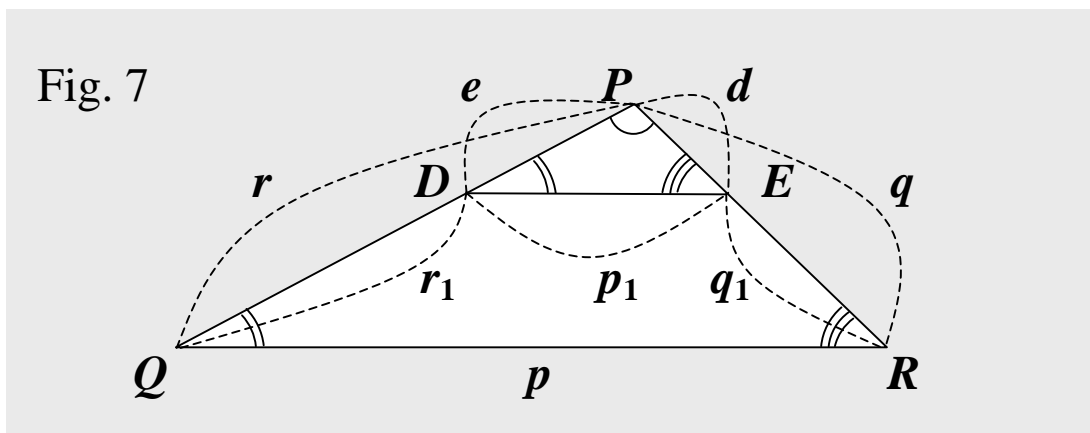
So again, if corresponding angles are equal, the two lines with the corresponding angles are parallel, and vice versa.

We often use the basic above solving problems in many kinds. If not sure of it, refer to the lessons called **Angles and Lines**, where you can learn how corresponding and alternate angles work.

Now, we know that  $\triangle MCY$  and  $\triangle ACG$  are similar.

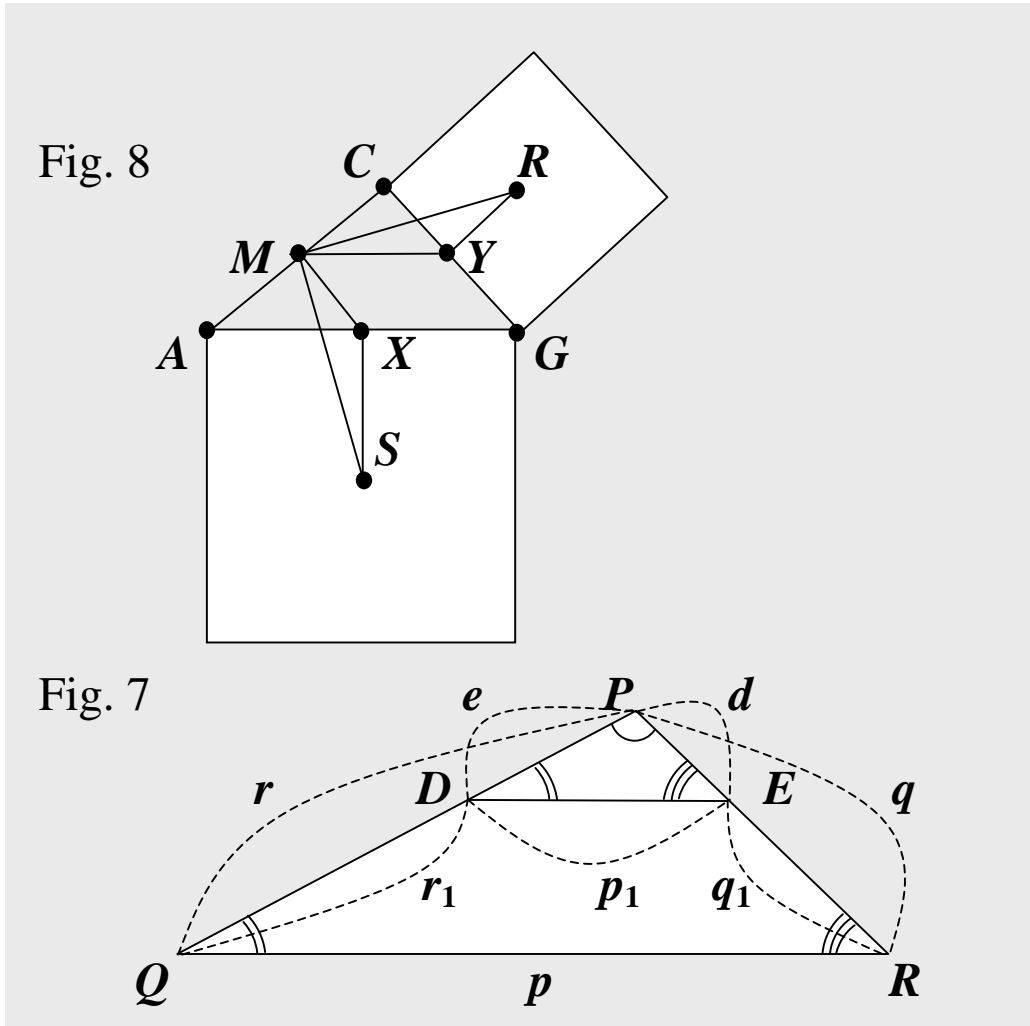


So both triangles have the same angle groups.  
And that's what the definition (2-1) below is saying.



$$(2-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \angle Q = \angle D$$

Note that  $\angle Q$  matches  $\angle CAG$ , and  $\angle D$  matches  $\angle CMY$ .



Thus,  $\angle CMY = \angle CAG$ , and these two angles are the same corresponding angles if we take  $AC$  as the transversal. So we get this:  $MY \parallel AG$ , since the two angles above are the same corresponding angles.

And  $XG$  is a part of  $AG$ , so we get this:  $MY \parallel XG$ .

By the same token, we get this:  $|MX| = |YG|$ , too.

And we get this:  $MX \parallel YG$ , too.

In this case,  $\triangle AXM$  is nested inside  $\triangle AGC$ , and matches  $\triangle PDE$ . And  $\triangle AGC$  matches  $\triangle PQR$ .

Fig. 8

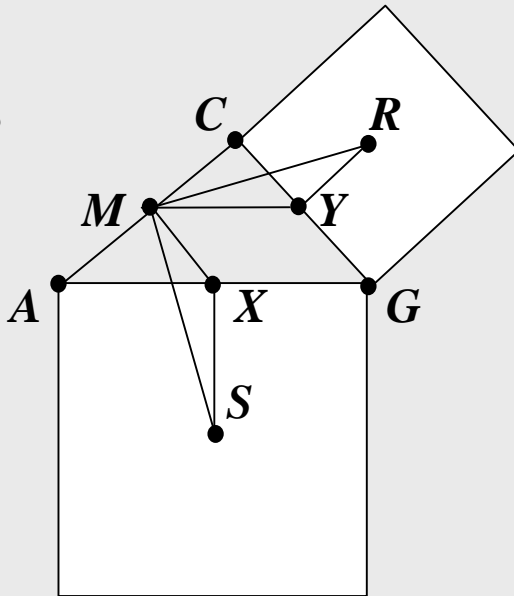
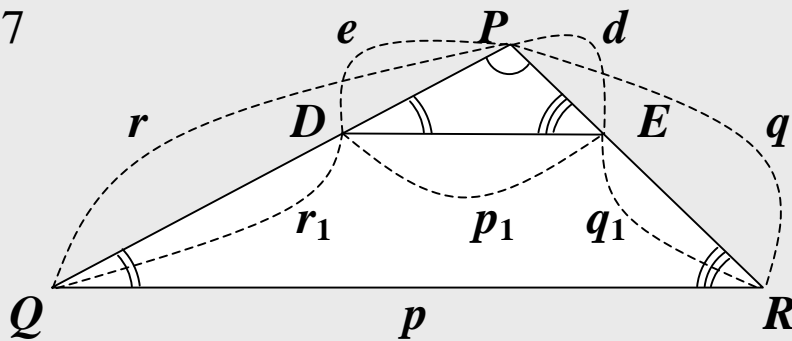
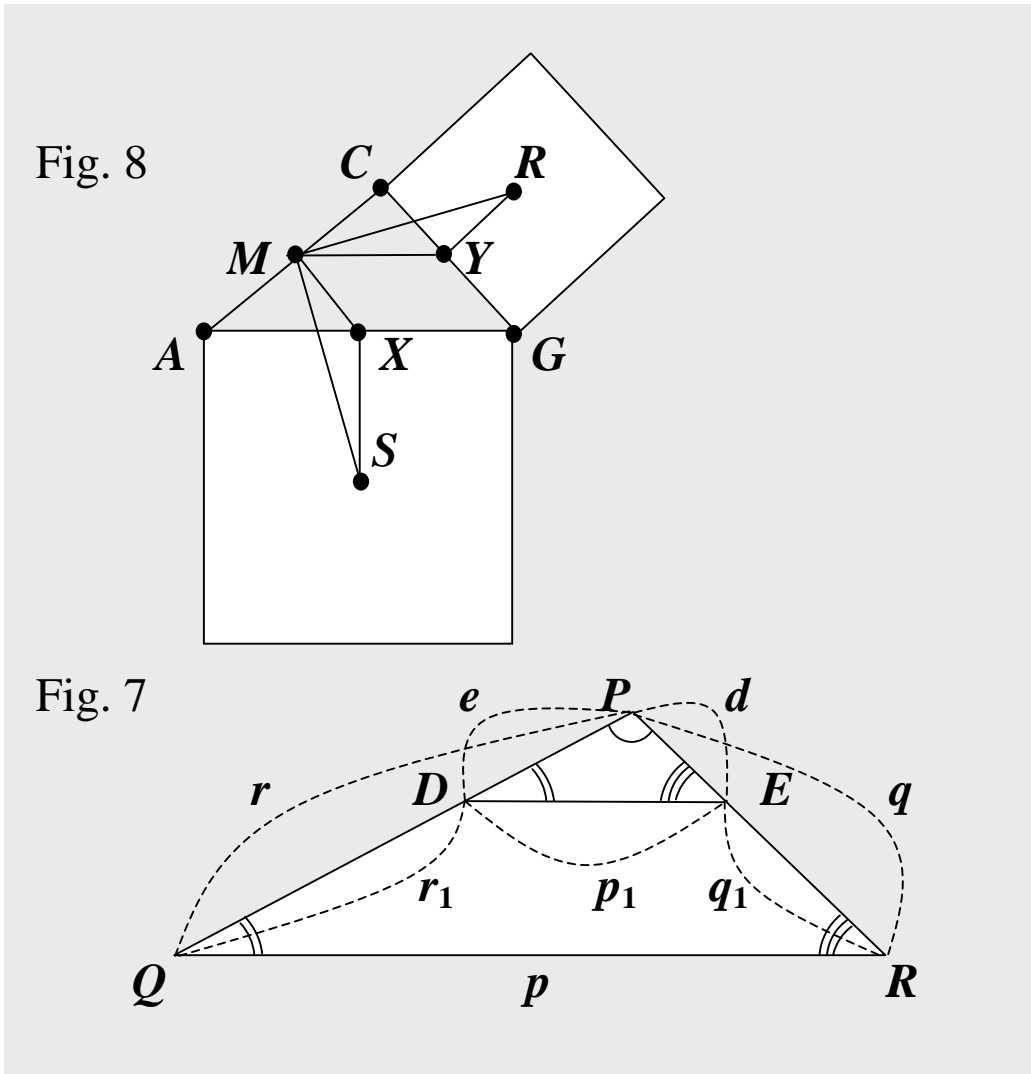


Fig. 7



This time, let's use the definition (3-1), and then, use (a-1).



$$(3-1) \quad \Delta PQR \sim \Delta PDE \Leftrightarrow \frac{e}{r_1} = \frac{d}{q_1} \text{ where } e \text{ matches } r_1$$

$AX$ ,  $r_1$  matches  $XG$ ,  $d$  matches  $AM$ , and  $q_1$  matches  $MC$ .

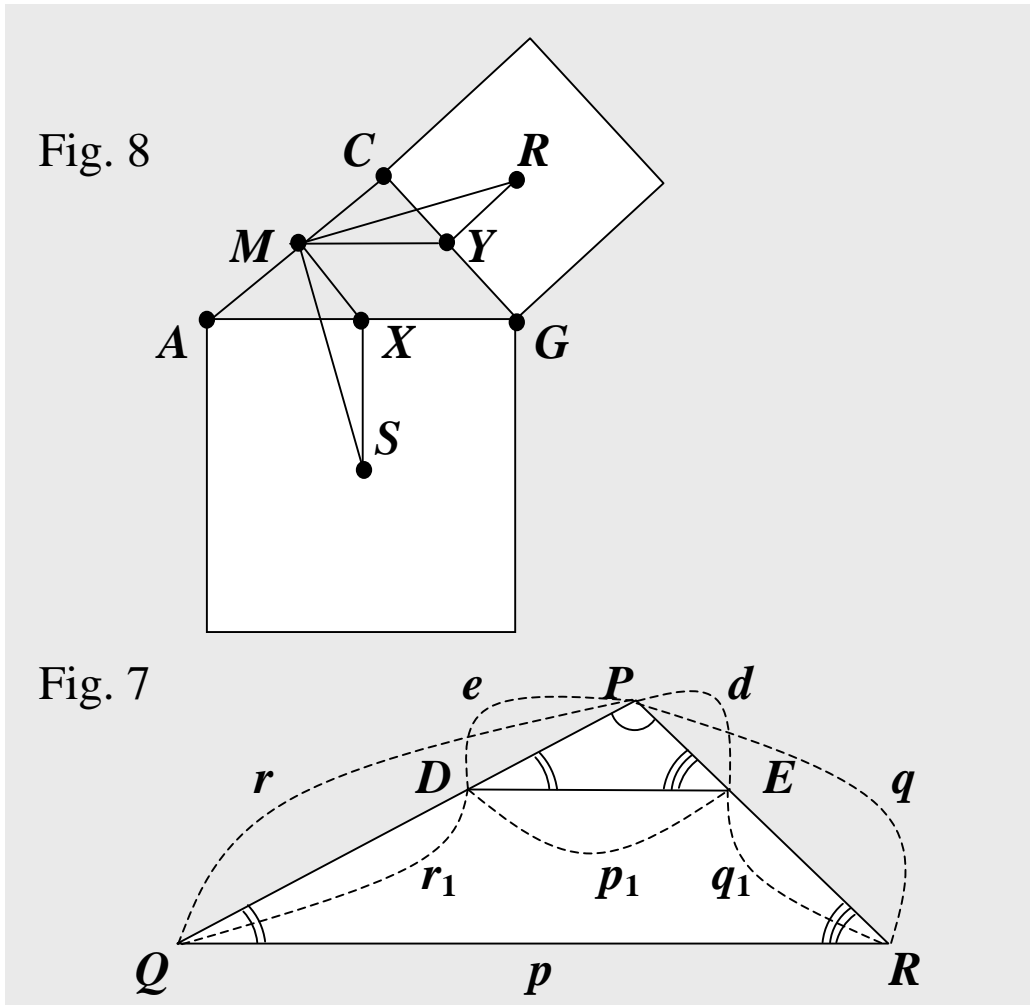
$$(a-1) \quad \frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}, \text{ three equal ratios, the same scale factor}$$



Checking thus, to see if  $\triangle AXM$  and  $\triangle AGC$  are similar using

(3-1), we get this:  $\frac{|AX|}{|XG|} = \frac{1}{1} = \frac{|AM|}{|MC|}$ , which shows that

they are indeed similar, since  $\angle A$  is common.



$$(3-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \frac{e}{r_1} = \frac{d}{q_1} \text{ where } e \text{ matches}$$

$AX$ ,  $r_1$  matches  $XG$ ,  $d$  matches  $AM$ , and  $q_1$  matches  $MC$ .

Therefore, using (a-1), we get this:  $\frac{|AM|}{|AC|} = \frac{1}{2} = \frac{|AX|}{|AG|}$ .

Fig. 8

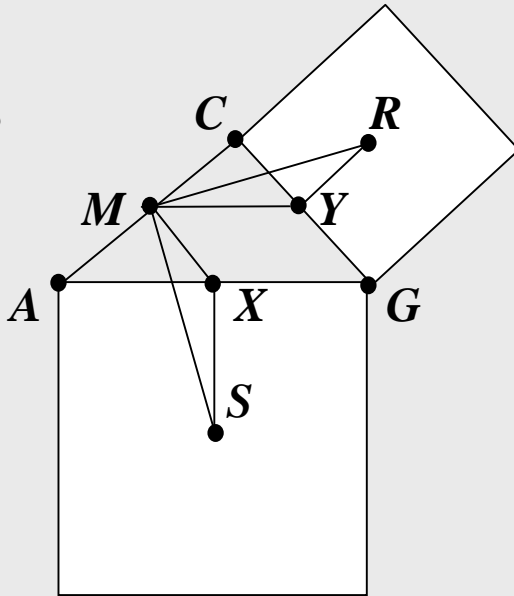
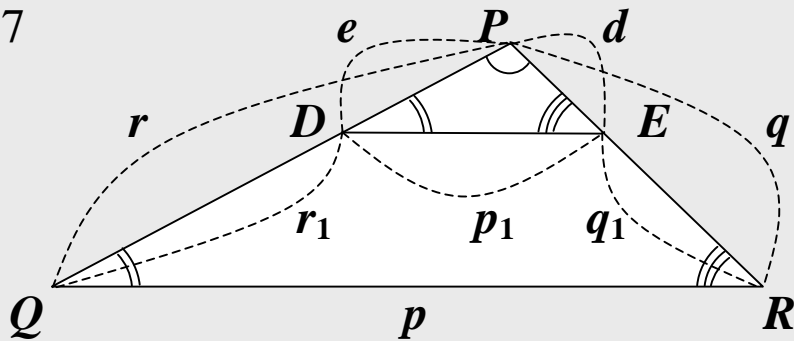


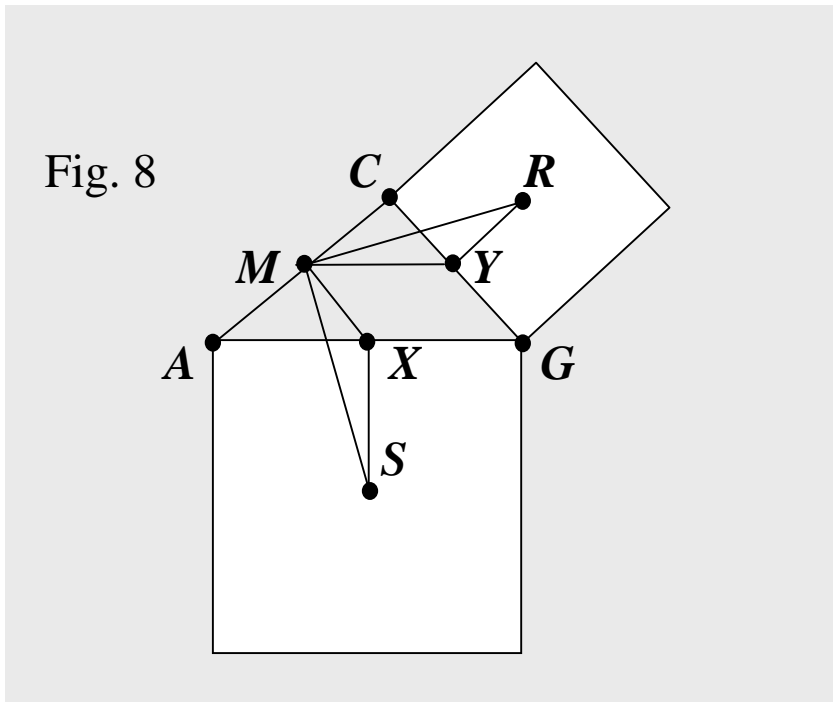
Fig. 7



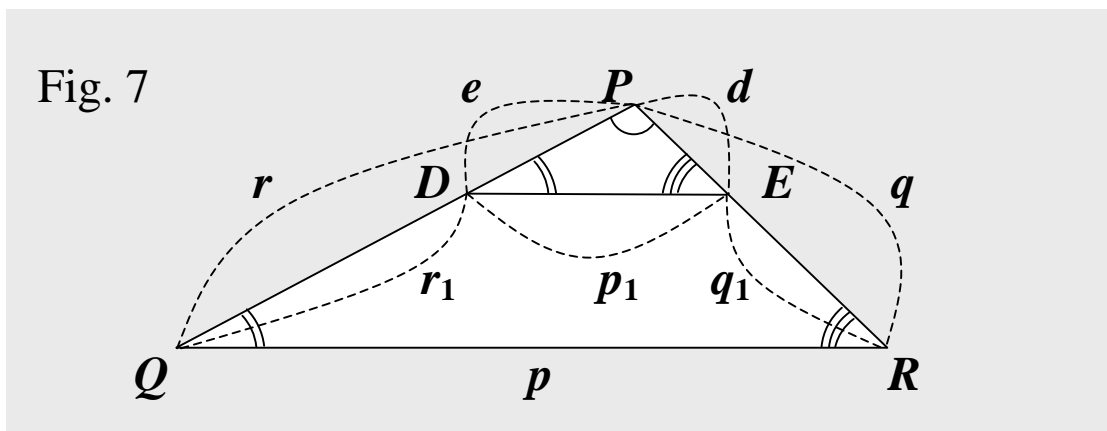
(a-1)  $\frac{p_1}{p} = \frac{d}{q} = \frac{e}{r}$ , equal ratios, the same scale factor

So we get this:  $\frac{|MX|}{|CG|} = \frac{1}{2} \Rightarrow |MX| = \frac{|CG|}{2} = |YG|$

So we get this:  $|MX| = |YG|$ .



And, of course, we get this, too:  $MX \parallel YG$ , because the two triangles  $\triangle AXM$  and  $\triangle AGC$  are similar. In the figure below, we have this:  $\triangle PDE \sim \triangle PQR$ , so we have this, too:  $p_1 \parallel p$ .

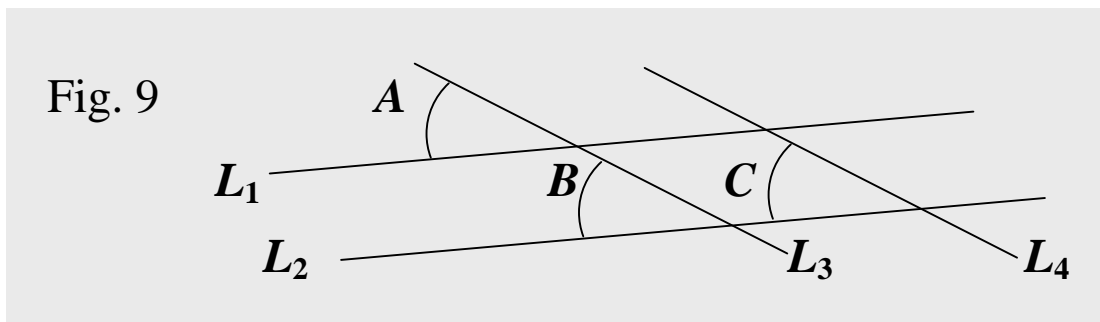


If however, answering again, the question, why parallel, we can say this again:

If the two lines with corresponding angles are parallel, the corresponding angles are equal, and vice versa.

So if corresponding angles are equal, the two lines with the corresponding angles are parallel.

For instance, in Fig. 9 below, if we have this:  $\angle A = \angle B$ , we get this:  $L_1 \parallel L_2$ , because the two angles  $\angle A$  and  $\angle B$  are corresponding angles, since  $L_3$  is the transversal.

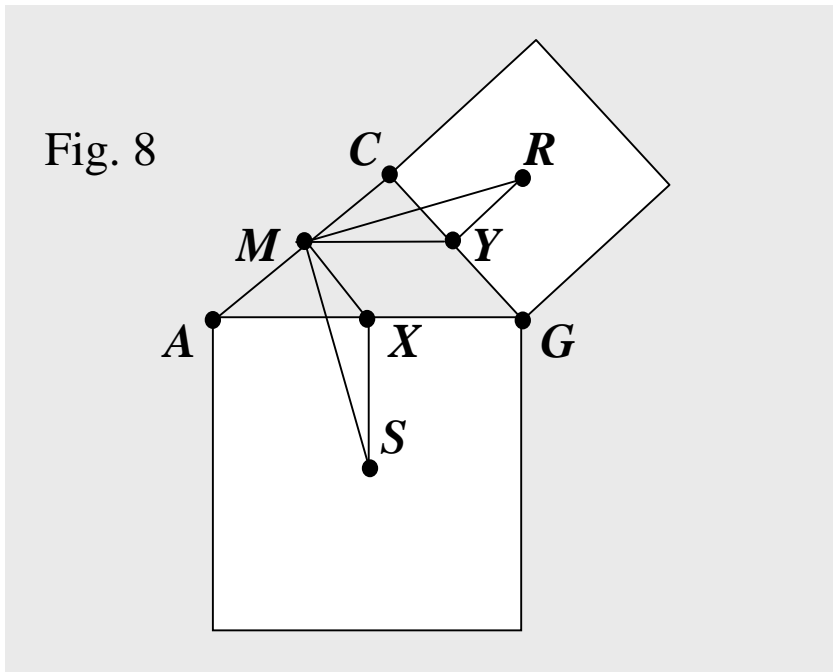


So again, if corresponding angles are equal, the two lines with the corresponding angles are parallel, and vice versa.

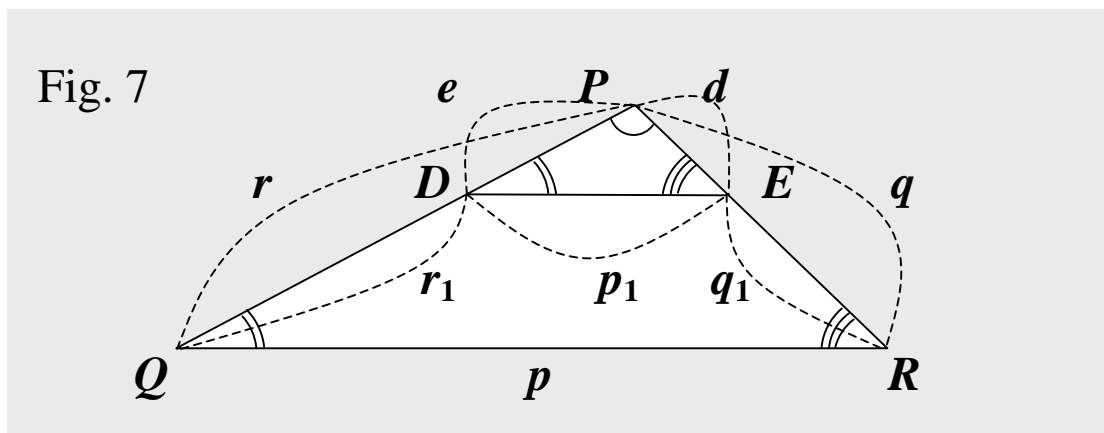
And we often use the basic above solving many problems.

So you may want know and keep it very well.

And now, we know that  $\triangle AXM$  and  $\triangle AGC$  are similar.



So both triangles have the same angle groups.  
And that's what the definition (2-1) below is saying.



$$(2-1) \quad \triangle PQR \sim \triangle PDE \Leftrightarrow \angle Q = \angle D$$

Note that  $\angle Q$  matches  $\angle AGC$ , and  $\angle D$  matches  $\angle AXM$ .

Fig. 8

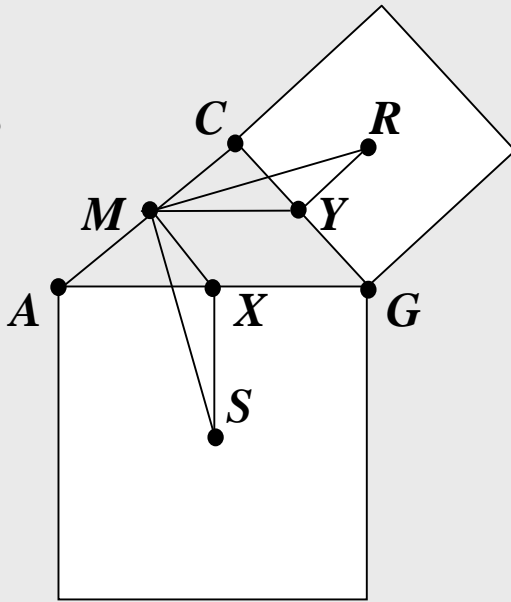
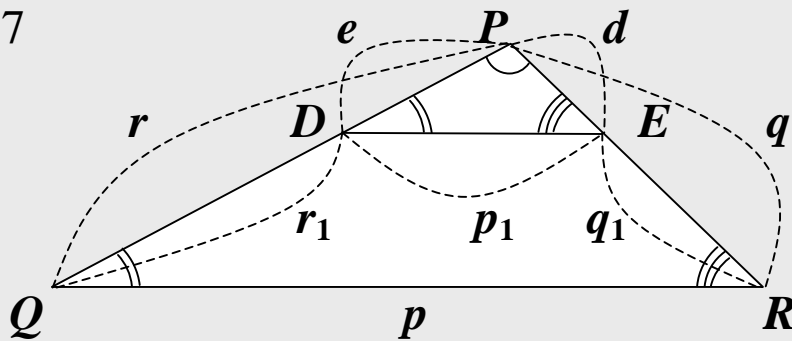


Fig. 7



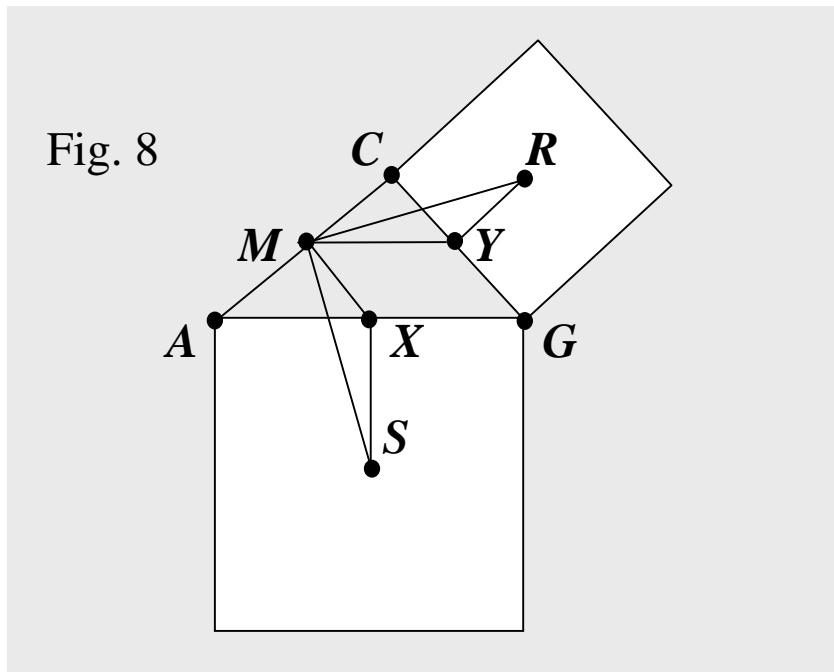
Thus,  $\angle AXM = \angle AGC$ , and these two angles are the same corresponding angles if we take  $AG$  as the transversal,.

So we get this:  $MX \parallel CG$ , since the two angles above are the same corresponding angles.

And  $YG$  is a part of  $CG$ , so, we get this:  $MX \parallel YG$ .

And summing up, for now, we've got these:

$$|MY| = |XG|, \quad MY \parallel XG, \quad |MX| = |YG|, \quad MX \parallel YG$$



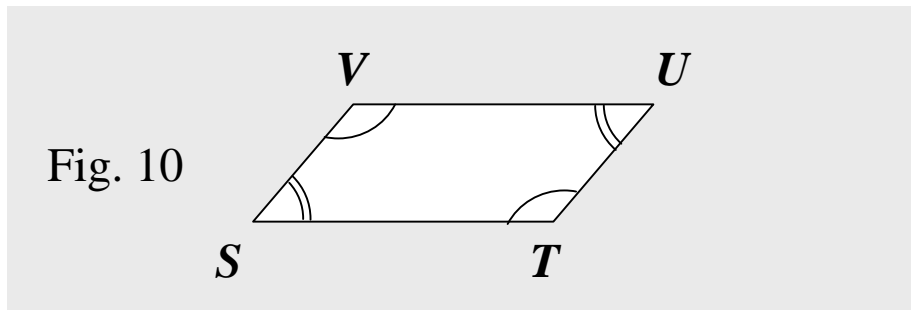
So the opposite sides are of the same length and parallel.

What then is the tetragon  $MXGY$ ?

It's a parallelogram, and its definition can be as follows.

A tetragon is a parallelogram if and only if each pair of opposite sides are parallel.

What then about the same lengths?



We can make another definition using the same lengths the way as follows.

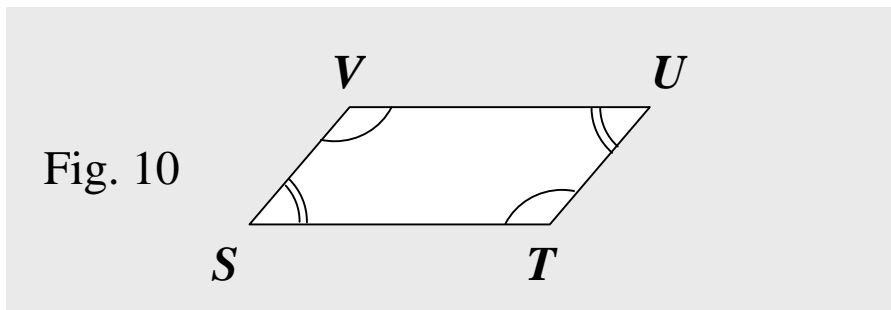
A tetragon is a parallelogram if and only if each pair of opposite sides have the same lengths.

And the conditions in the definitions are called properties of a parallelogram, too. So the properties are as follows.

Each pair of opposite sides are parallel.

And the two sides in each pair have the same lengths.

Solving thus, problems with a parallelogram, we can use the properties above. And that's not it. For example, each pair of angles facing each other are the same, and every pair of angles next to each other add up to  $180^\circ$ .



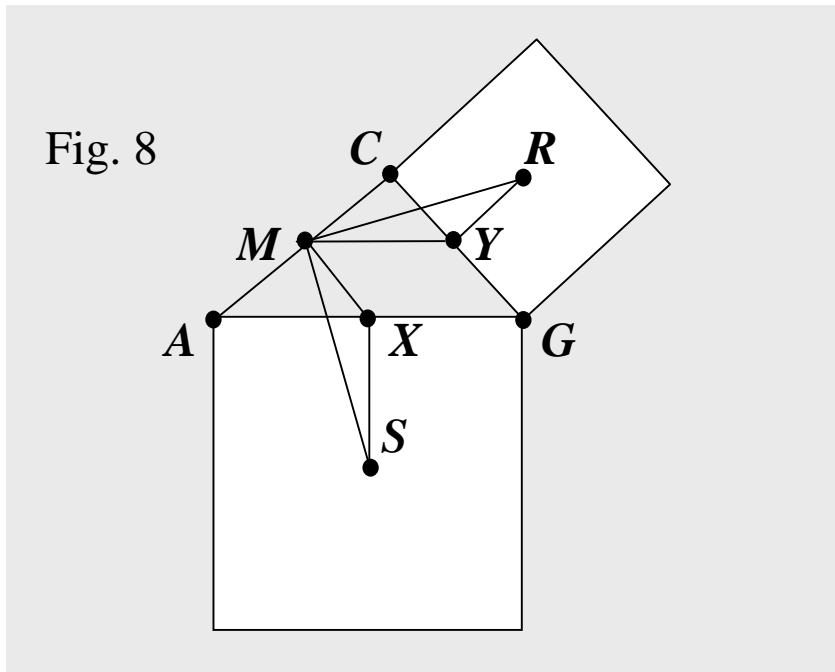
So in Fig. 10 above, we get these:  $\angle S = \angle U$  and  $\angle V = \angle T$ , and also, this:

$$\angle S + \angle T = \angle T + \angle U = \angle U + \angle V = \angle V + \angle S = 180^\circ.$$

It has other properties, too. And we have a lot to say about parallelograms. If you want to see more about the math idea called a parallelogram, you may want to refer to ***Parallelograms*** in the lesson notes called ***Basic Shapes***.

So where are we now?

Been taking steps to prove that  $\triangle SMR$  is an isosceles right triangle. We are not there yet. And we are now looking at a tetragon  $MXGY$ , which has turned out to be a parallelogram.

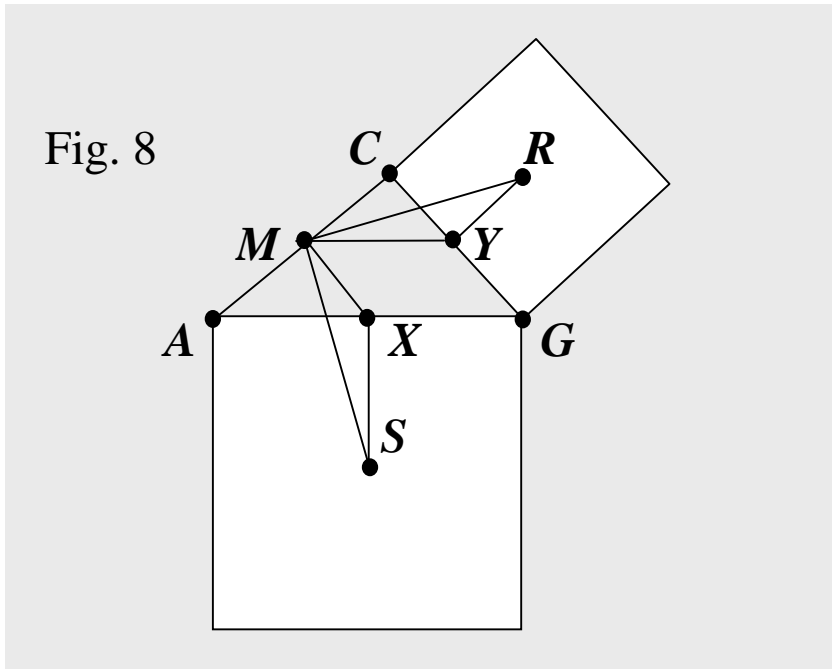


And we've covered a lot.

So let's now recapitulate what's been covered, so that we can see what we can work with and what could be done in the next step.

First, since the two points  $S$  and  $R$  are the centers of the squares, we've got these:

$$SX \perp XG, \quad |SX| = |XG|, \quad RY \perp YG, \quad |RY| = |YG|$$



Next, the tetragon  $MXGY$  turned out to be a parallelogram, which means, we've got these, too:

$$|MY| = |XG|, \quad MY \parallel XG, \quad |MX| = |YG|, \quad MX \parallel YG$$

So now, putting things together, we get these:

$$SX \perp XG, \quad |SX| = |XG|, \quad RY \perp YG, \quad |RY| = |YG|$$

$$|MY| = |XG|, \quad MY \parallel XG, \quad |MX| = |YG|, \quad MX \parallel YG$$

And simplifying some of these:

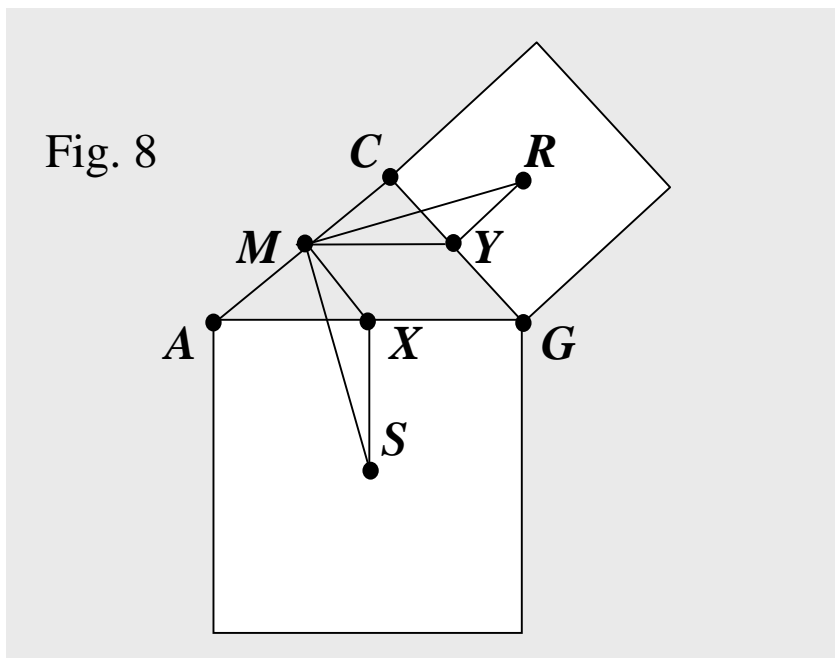
$$SX \perp XG, \quad |SX| = |XG|, \quad RY \perp YG, \quad |RY| = |YG|$$

$$|MY| = |XG|, \quad MY \parallel XG, \quad |MX| = |YG|, \quad MX \parallel YG$$

we can get these:

$$SX \perp XG, \quad RY \perp YG, \quad MY \parallel XG, \quad MX \parallel YG,$$

$$|SX| = |XG| = |MY|, \quad |RY| = |YG| = |MX|$$

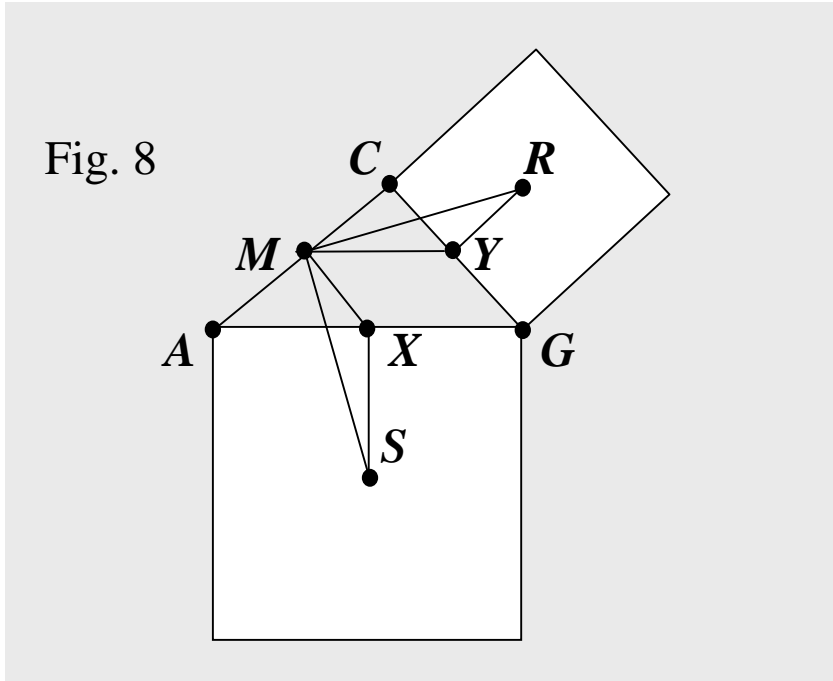


So now, looking at closely  $\triangle SMX$  and  $\triangle RMY$ , together with

these two:  $|SX| = |XG| = |MY|$  and  $|RY| = |YG| = |MX|$ ,

what then can you notice?

We can notice that  $\triangle SMX$  and  $\triangle RMY$  share two sides.



What then are the two sides?

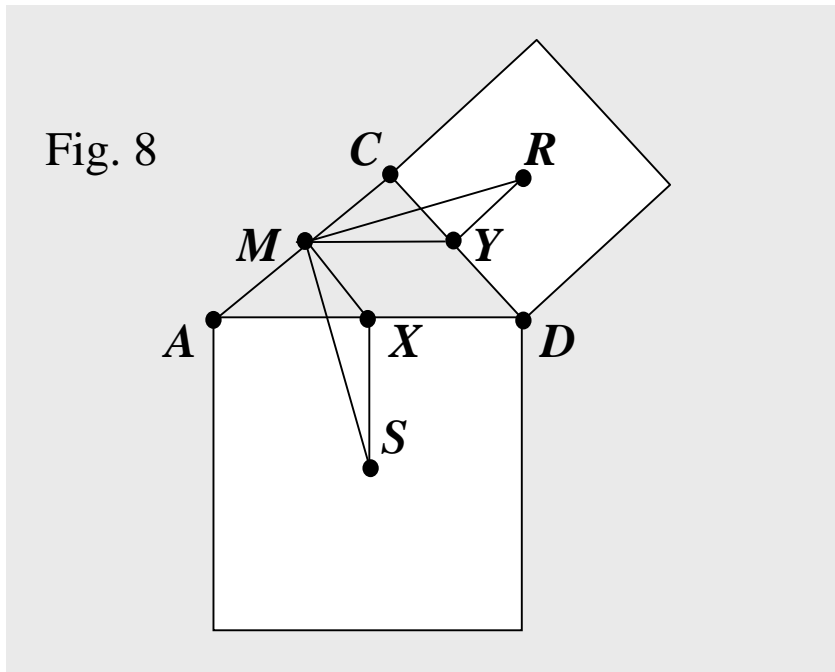
We have these:  $|SX| = |XG| = |MY|$  and  $|RY| = |YG| = |MX|$ ,

so we can get these two:  $|SX| = |MY|$  and  $|RY| = |MX|$ .

That is to say that in  $\triangle SMX$  and  $\triangle RMY$ , the side  $SX$  is the same as the side  $MY$ , and  $RY$  is the same as  $MX$ .

So what can we expect?

We can expect that  $\Delta SMX$  and  $\Delta RMY$  are identical; that is, we can expect this:  $\Delta SMX \equiv \Delta RMY$ , which is now thus, a hypothesis to be proven.



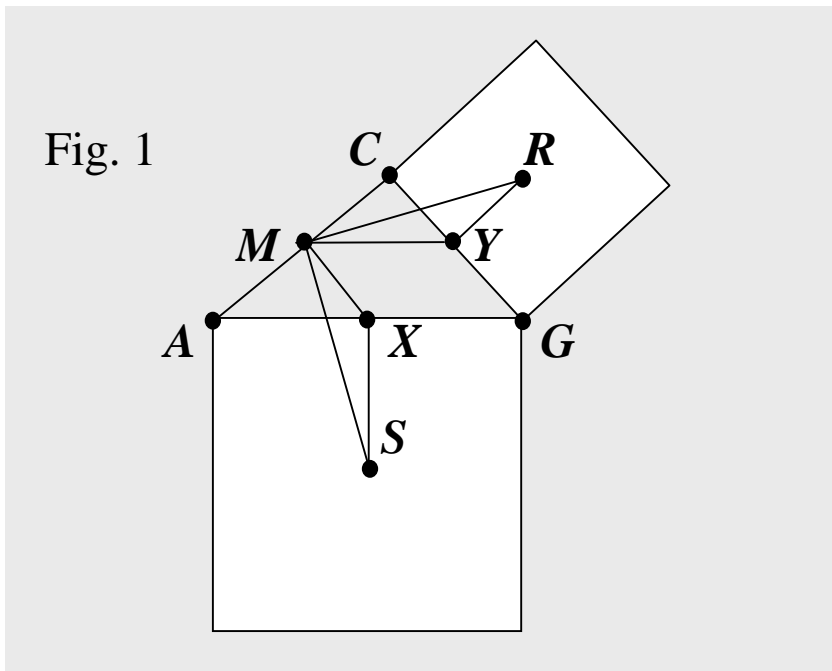
So if proving this:  $\Delta SMX \equiv \Delta RMY$ , we can say that the two line segments  $SM$  and  $RM$  have the same lengths; that is, we get this:  $|SM| = |RM|$ , which means that  $\Delta SMR$  is an isosceles triangle.

Proving thus, this identity:  $\Delta SMX \equiv \Delta RMY$ , and also, this:  $\angle SMR = 90^\circ$ , we prove that  $\Delta SMR$  is an isosceles right triangle. We'll do those two proofs in the next lesson.

## Van Aubel's Theorem Part 5

First, in the previous lesson, since  $S$  and  $R$  are the centers of the squares, we've got these:

$$SX \perp XG, \quad |SX| = |XG|, \quad RY \perp YG, \quad |RY| = |YG|$$



Next, the tetragon  $MXGY$  turned out to be a parallelogram, which means, we've got these, too:

$$|MY| = |XG|, \quad MY \parallel XG, \quad |MX| = |YG|, \quad MX \parallel YG$$

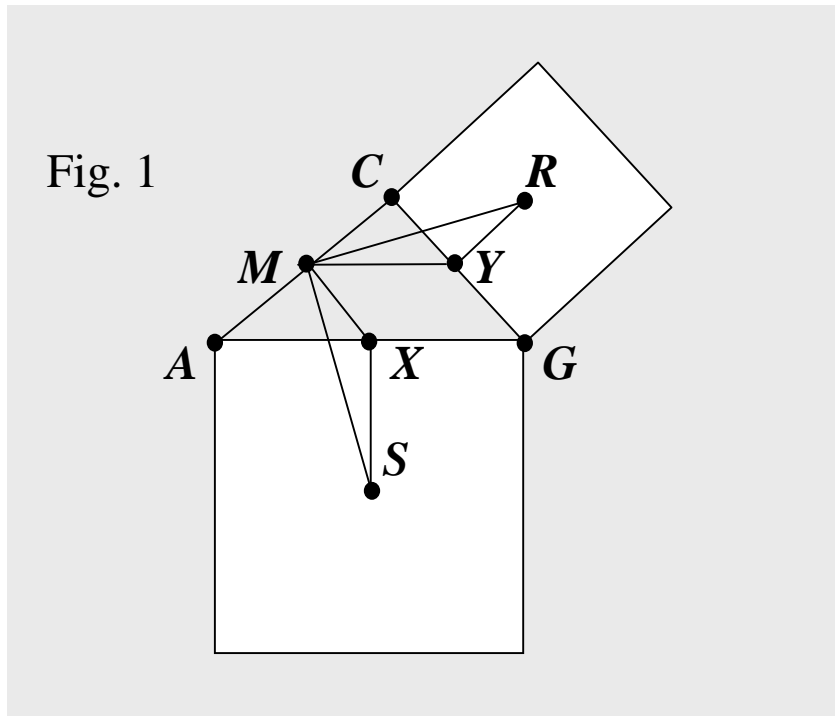
And putting things together simplifying some of those above, we can get these:  $SX \perp XG$ ,  $RY \perp YG$ ,  $MY \parallel XG$ ,

$$MX \parallel YG, \quad |SX| = |XG| = |MY|, \quad |RY| = |YG| = |MX|$$



Let's now begin with this equality:  $|SM| = |RM|$ , which is saying that the two sides have the same lengths.

So what do we have now to work with to prove the equality?



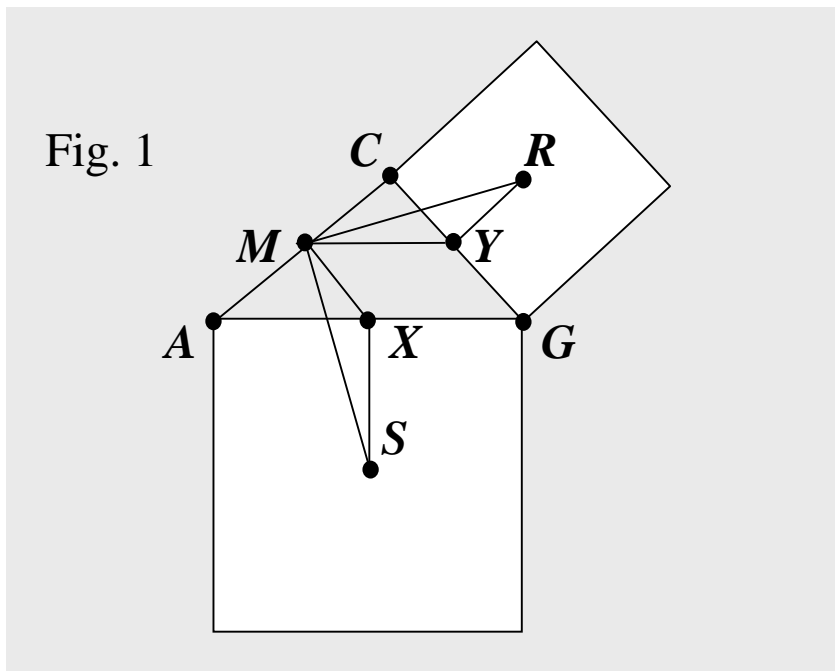
Looking at closely the two triangles  $\Delta SMX$  and  $\Delta RMY$ , we can notice that  $SM$  is a side of  $\Delta SMX$  and that  $RM$  is a side of  $\Delta RMY$ . And we have these findings:

$$SX \perp XG, \quad RY \perp YG, \quad MY \parallel XG, \quad MX \parallel YG,$$

$$|SX| = |XG| = |MY|, \quad |RY| = |YG| = |MX|$$

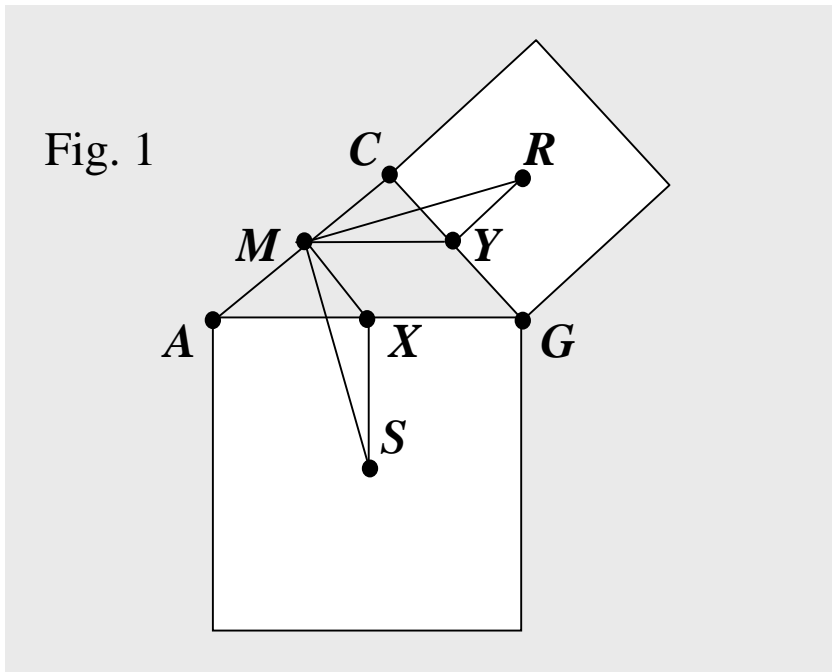
So among those findings above, looking at these two findings:  $|SX| = |XG| = |MY|$  and  $|RY| = |YG| = |MX|$ , we can notice that the two triangles  $\Delta SMX$  and  $\Delta RMY$  share two sides. What then are the two?

From these:  $|SX| = |XG| = |MY|$  and  $|RY| = |YG| = |MX|$ , we can get these:  $|SX| = |MY|$  and  $|RY| = |MX|$ .



So we can say that  $\Delta SMX$  and  $\Delta RMY$  share two sides, which are these:  $SX$  and  $MX$ , or these two:  $RY$  and  $MY$ .

So we can expect that  $\triangle SMX$  and  $\triangle RMY$  are identical; that is, we can expect this:  $\triangle SMX \cong \triangle RMY$ , which is now thus, a hypothesis to be proven.



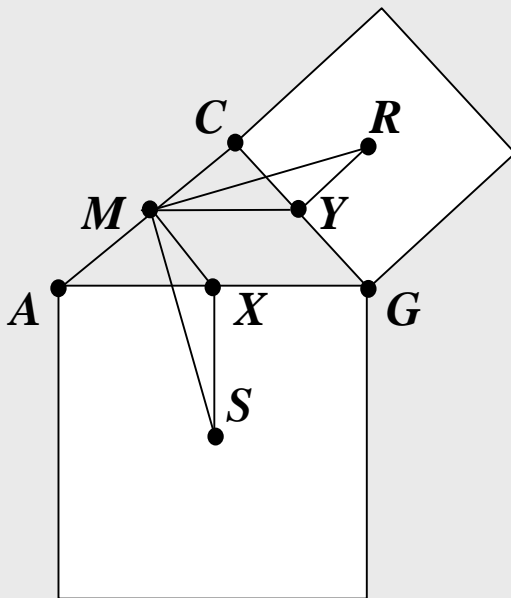
So if proving this:  $\triangle SMX \cong \triangle RMY$ , we can say that the two line segments  $SM$  and  $RM$  are of the same length; that is, we get this:  $|SM| = |RM|$ , which means that  $\triangle SMR$  is an isosceles triangle.

Proving thus, this identity:  $\triangle SMX \cong \triangle RMY$ , and also this:  $\angle SMR = 90^\circ$ , we prove that  $\triangle SMR$  is an isosceles right triangle.

We'll do the proof of this identity first:  $\Delta PXM \equiv \Delta QYM$ . And doing it, we can use a basic, which is saying this:

If two triangles share an angle and the two sides making the angle, the two triangles are identical, and vice versa.

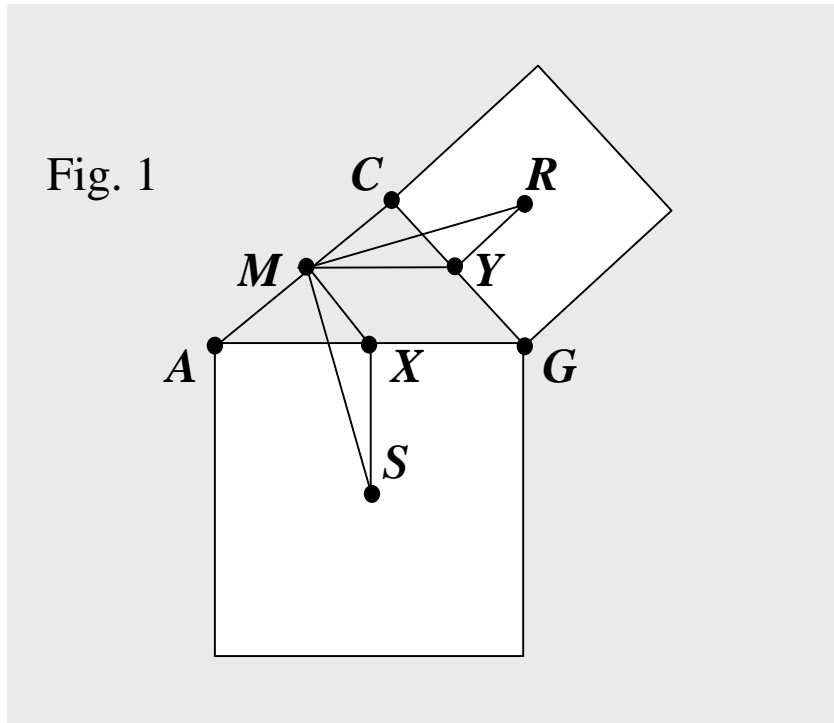
Fig. 1



So if we prove this:  $\angle MXS = \angle RYM$ , we prove this identity:  $\Delta SMX \equiv \Delta RMY$ , since the triangles share two sides making this angle:  $\angle MXS$ , which equals this:  $\angle RYM$ .

So let's see now, how we can get this:  $\angle MXS = \angle RYM$ .

First off, we have these:  $SX \perp XG$ , and  $RY \perp YG$ .



That is, we have these:  $SX \perp AX$ , and  $RY \perp CY$ .

So we get this:  $\angle AXS = 90^\circ = \angle CYR$ . Thus, we get these:

$\angle MXS = 90^\circ + \angle AXM$ , and  $\angle RYM = 90^\circ + \angle MYC$ .

So in the next step, what do we need to get?

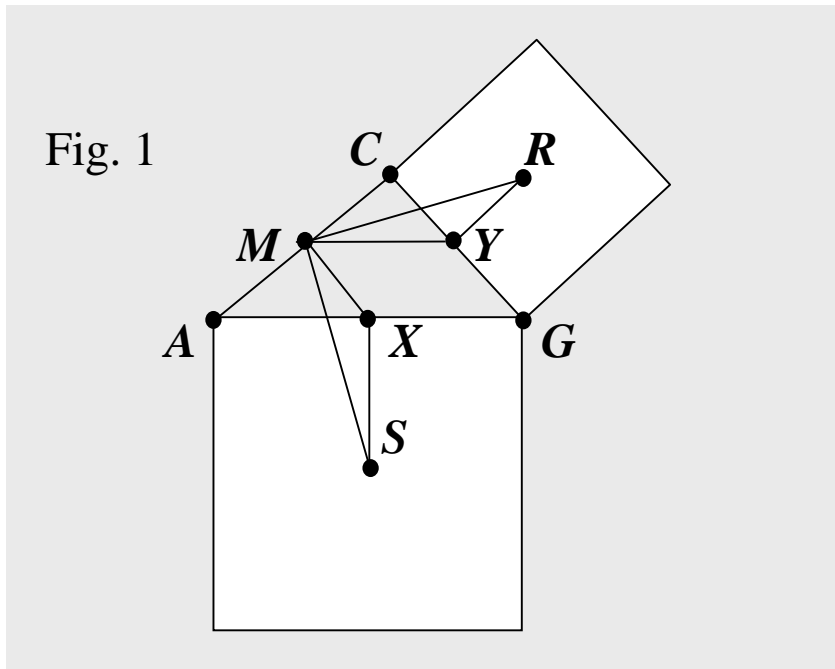
If  $\angle AXM = \angle MYC$ , we get this:  $\angle MXS = \angle RYM$ .

So getting this:  $\angle AXM = \angle MYC$ , we get the proof.

We are going to use two ways to get it, and one is this:

We have these:  $\triangle MCY \sim \triangle ACG$ , and  $\triangle AMX \sim \triangle ACG$ . So

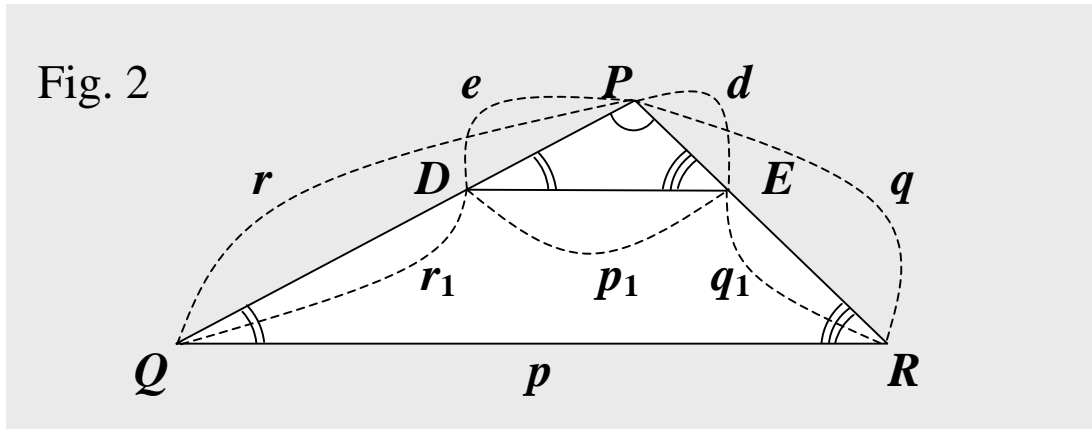
we get these:  $MY \parallel AG$ , and  $MX \parallel CG$ . Why, though?



First,  $\triangle MCY$  is nested inside, and is similar to  $\triangle ACG$ , and the same is true of  $\triangle AMX$ , too.

Next, as mentioned in one of the previous lessons, if you get to see this kind of situation where **two triangles are nested and similar**, you can directly use a fact that **the two sides away from each other are parallel**, as in the case of the two sides  $p_1$  and  $p$  in the figure as follows.

In the figure below, we have this:  $\angle Q = \angle D$ , which means that corresponding angles are equal.



$$(2-1) \quad \Delta PQR \sim \Delta PDE \Leftrightarrow \angle Q = \angle D$$

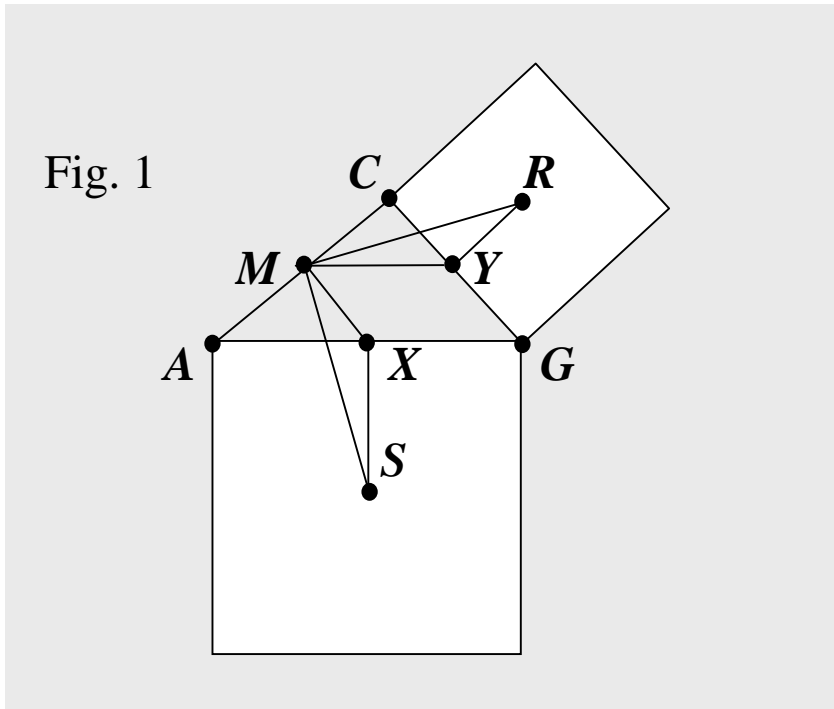
It's because  $\angle Q$  and  $\angle D$  are corresponding angles if we take the side  $PQ$  as the transversal.

And we've covered the fact that if corresponding angles are equal, the two lines with the corresponding angles are parallel, and vice versa.

So we get this:  $p_1 // p$ , that is,  $p_1$  and  $p$  are parallel.

Thus, if you get to see this kind of situation where **two triangles are nested and similar**, you can directly use the fact that **the two sides away from each other are parallel**.

So now, we have these:  $MY \parallel AG$ , and  $MX \parallel CG$ .



Taking, therefore, the side  $AG$  as the transversal, we get this:  $\angle AXM = \angle AGC$ , since these two angles are corresponding angles with parallel lines.

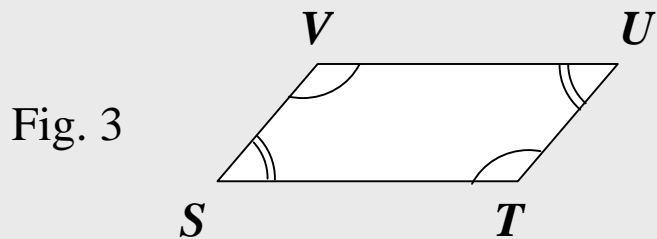
And taking the side  $CG$  as the transversal, we get this:  $\angle AGC = \angle MYC$ , since these angles are corresponding angles with parallel lines, too. So summing up, we get this:

$$\angle AXM = \angle AGC = \angle MYC, \text{ that is, } \angle AXM = \angle MYC.$$

Thus, the proof is done.

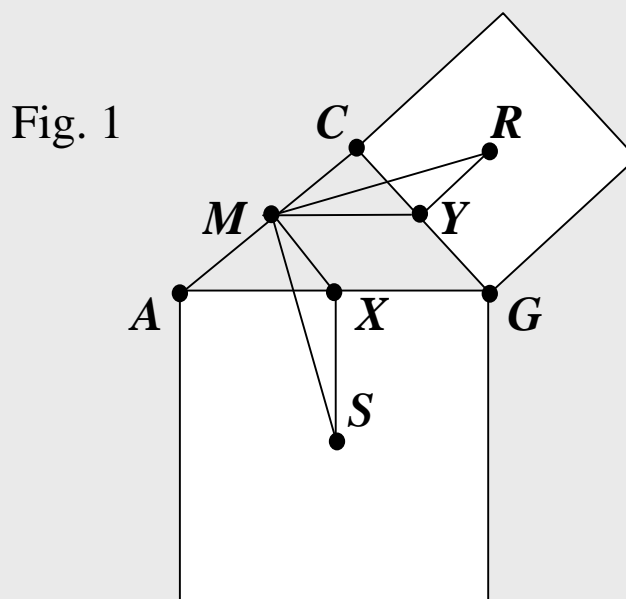
And of the two ways, the other is that we use this:

In a parallelogram, each pair of angles facing each other over its center are the same.



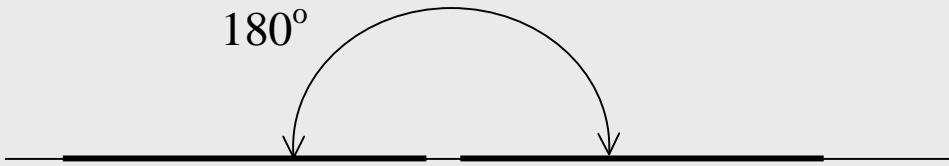
So if the quadrangle  $STUV$  above is a parallelogram, we get these:  $\angle S = \angle U$ , and  $\angle V = \angle T$ .

Thus, we get this:  $\angle MXG = \angle MYG$ .



Next, we call the angle of  $180^\circ$  a straight angle, because if two line segments are in a line, the angle between the two is  $180^\circ$  as shown below.

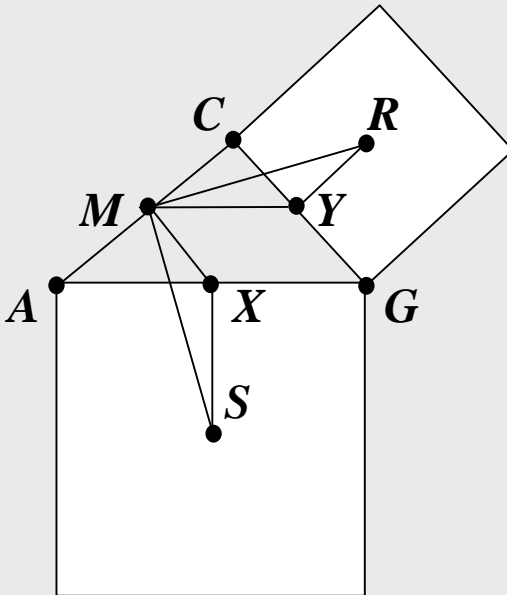
Fig. 4



So using the basic above, we can get this:

$$\angle AXM + \angle MXG = 180^\circ = \angle MYC + \angle MYG.$$

Fig. 1

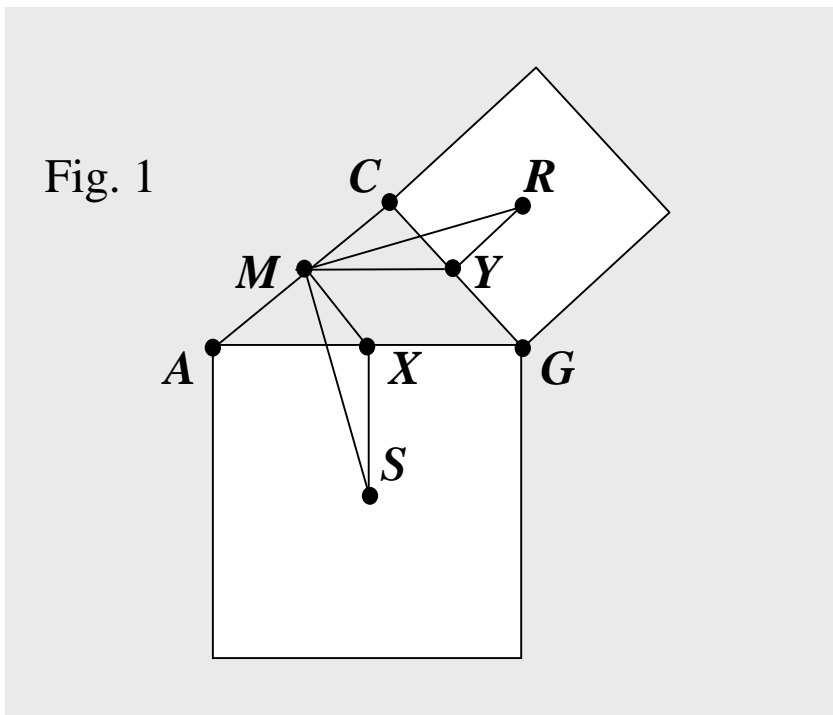


So we now have these:

$$\angle AXM + \angle MXG = 180^\circ = \angle MYC + \angle MYG.$$

$$\angle MXG = \angle MYG.$$

And  $\angle MXG$  and  $\angle MYG$  are in the first equality above.



So we get this:  $\angle AXM + \angle MXG = \angle MYC + \angle MXG$ .

Thus, we get this:  $\angle AXM = \angle MYC$ . And we've already got

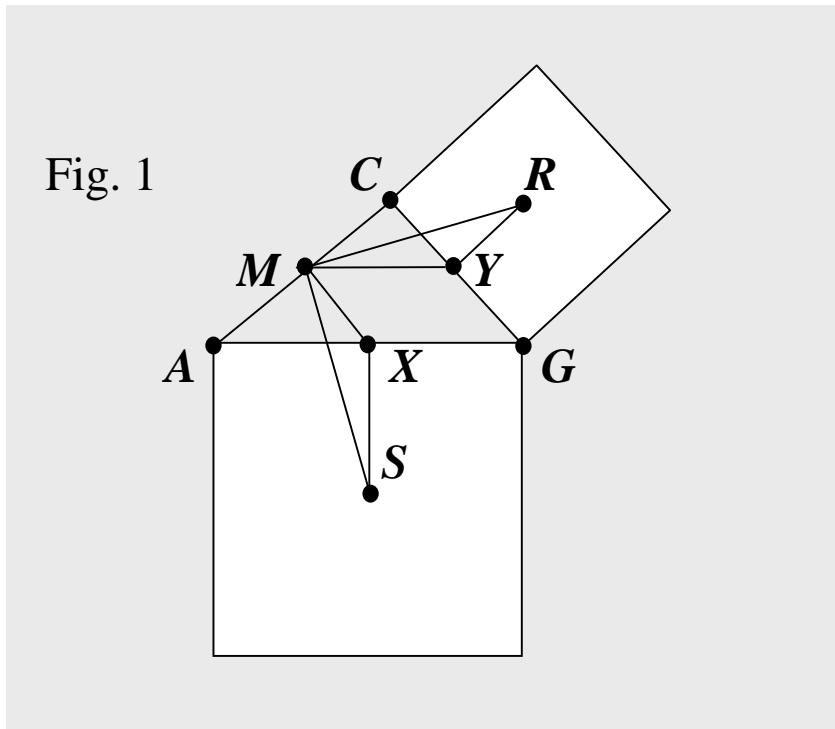
this:  $\angle ASX = 90^\circ = \angle RYC$ . So we've got these:

$$\angle MXS = 90^\circ + \angle AXM, \text{ and } \angle RYM = 90^\circ + \angle MYC.$$

So we now have these:

$$\angle MXS = 90^\circ + \angle AXM, \text{ and } \angle RYM = 90^\circ + \angle MYC.$$

$$\angle AXM = \angle MYC.$$

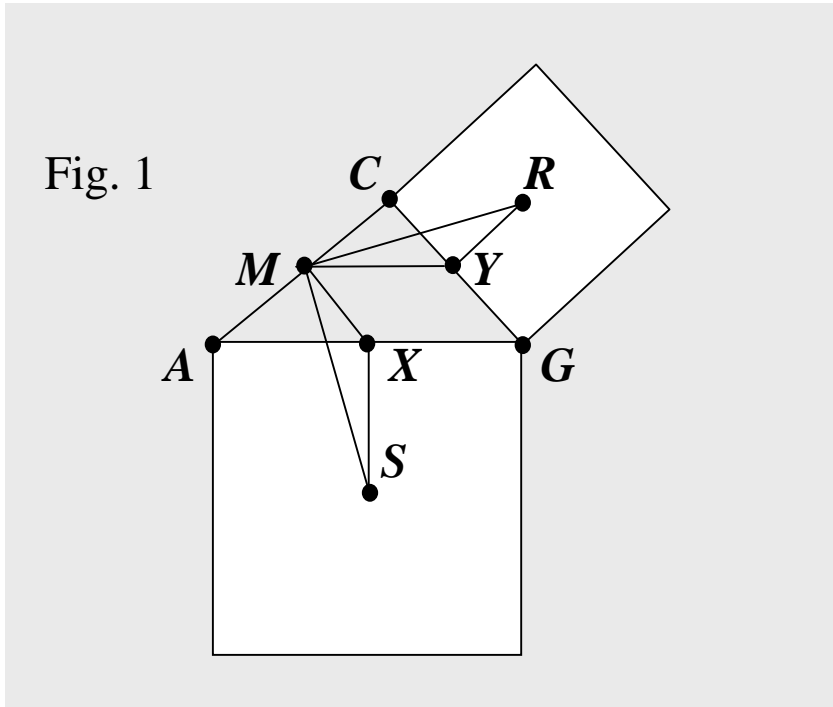


So we get these:

$$\angle MXS = 90^\circ + \angle AXM, \text{ and } \angle RYM = 90^\circ + \angle MYC.$$

Thus, we get this:  $\angle MXS = \angle RYM$ , and its proof is done.

Now, we have this:  $\angle MXS = \angle RYM$ .



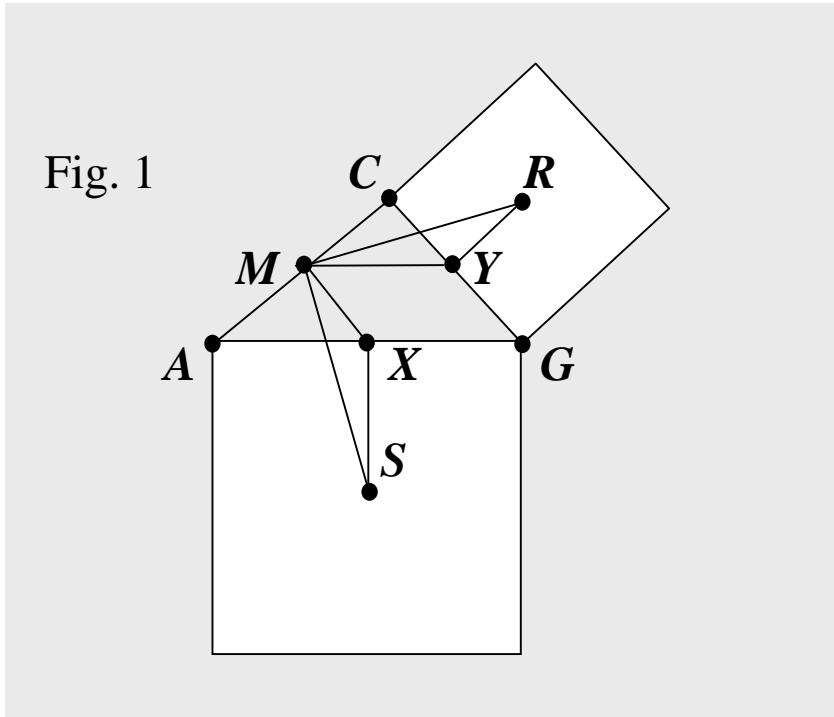
Also, we have these:  $|SX| = |MY|$ , and  $|RY| = |MX|$ .

And, as stated above, we have the basic that if two triangles share two sides and the angle between the two sides, the two triangles are identical.

So we can now say that the two triangles  $\triangle SMX$  and  $\triangle RMY$  are exactly the same, that is, identical.

That is, we have this:  $\triangle SMX \cong \triangle RMY$ .

Thus, we get this:  $|SM| = |RM|$ .

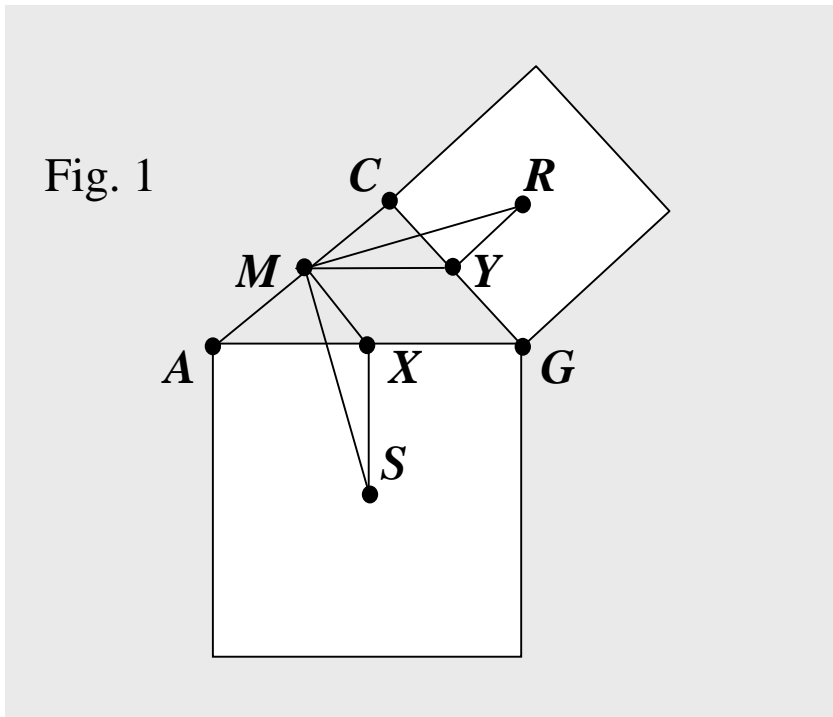


What then about the angle between  $SM$  and  $RM$ ?

It is  $90^\circ$ , that is,  $\angle SMR = 90^\circ$ , which needs to be proven.

So let's see now how we can get this:  $\angle SMR = 90^\circ$ .

To begin with, we have a basic that says this: in a triangle, all the three angles add up to  $180^\circ$ , so looking at  $\triangle MYR$ , we can get this:  $\angle YMR + \angle MRY + \angle MYR = 180^\circ \dots (1)$



And we have this:  $\angle MYR = \angle CYM + 90^\circ$ , since we have this:  $RY \perp CG$ , that is,  $\angle CYR = 90^\circ$ .

So we can put (1) above this way:

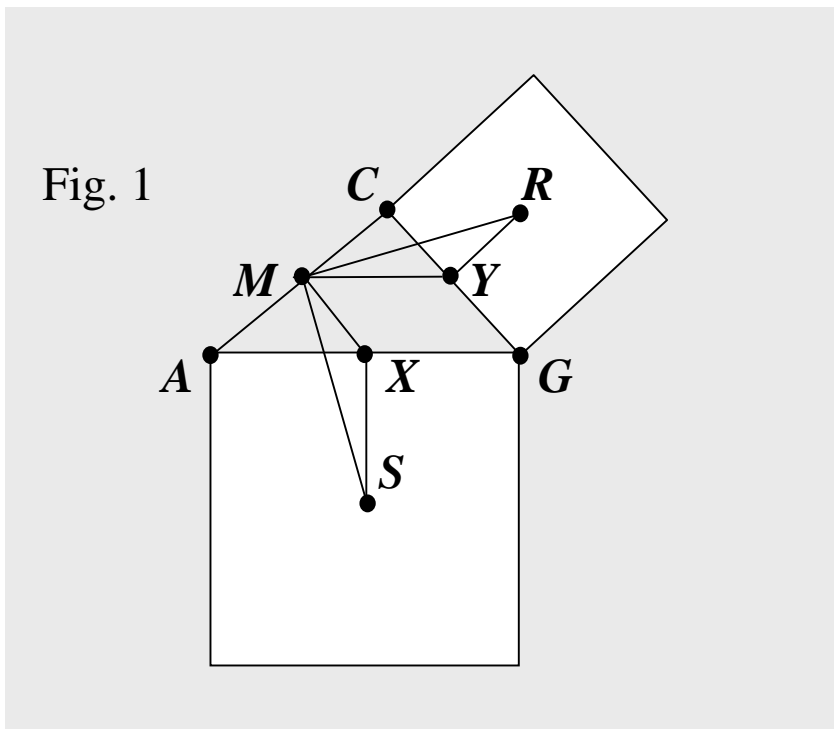
$$\angle YMR + \angle MRY + (\angle CYM + 90^\circ) = 180^\circ$$

Thus, we get this:  $\angle YMR + \angle MRY + \angle CYM = 90^\circ$ .

So now, we have this:

$$\angle YMR + \angle MRY + \angle CYM = 90^\circ \dots (2)$$

And we know this:  $\angle MRY = \angle XMS \dots (3)$



Thus, putting (3) into (2) above, we get this:

$$\angle YMR + \angle XMS + \angle CYM = 90^\circ.$$

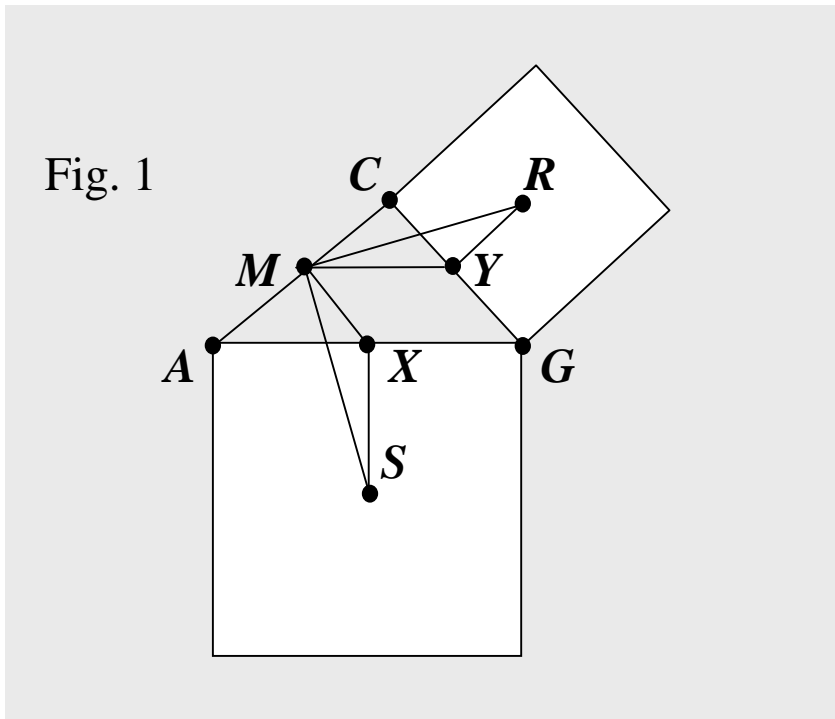
Next, we have this:  $MX \parallel CG$ .

So we get this:  $\angle CYM = \angle YMX$ .

So now, we have these:

$$\angle YMR + \angle XMS + \angle CYM = 90^\circ \dots (4)$$

$$\angle CYM = \angle YMX \dots (5)$$



Thus, putting (5) into (4) above, we get this:

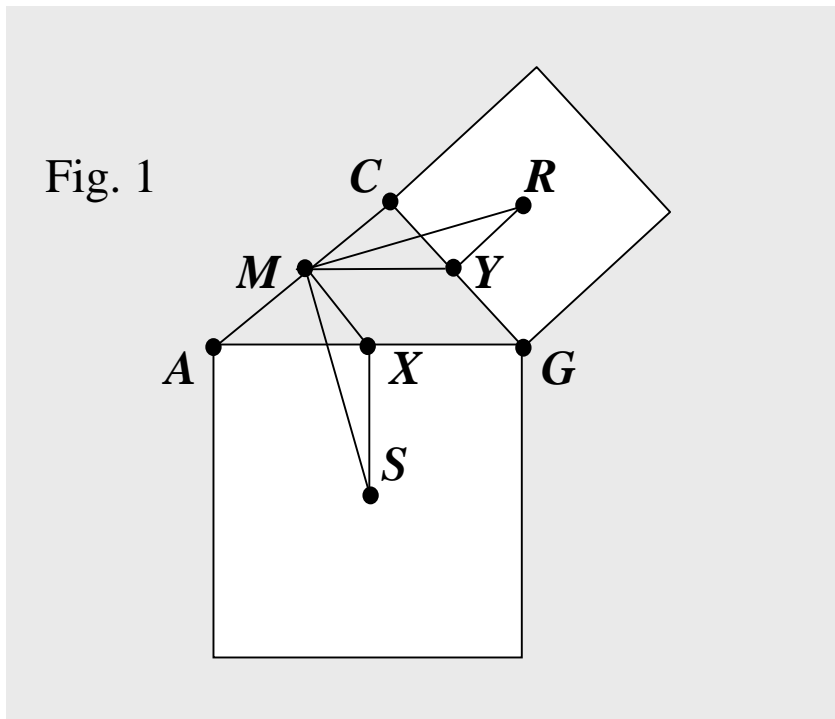
$$\angle YMR + \angle XMS + \angle YMX = 90^\circ.$$

And we know this:  $\angle SMR = \angle YMR + \angle XMS + \angle YMX$ .

Thus, we get this:  $\angle SMR = 90^\circ$ , and its proof is done.

Besides, the two triangles  $\triangle SMX$  and  $\triangle RMY$  are identical.

Now, we have this:  $\angle SMR = 90^\circ$ .



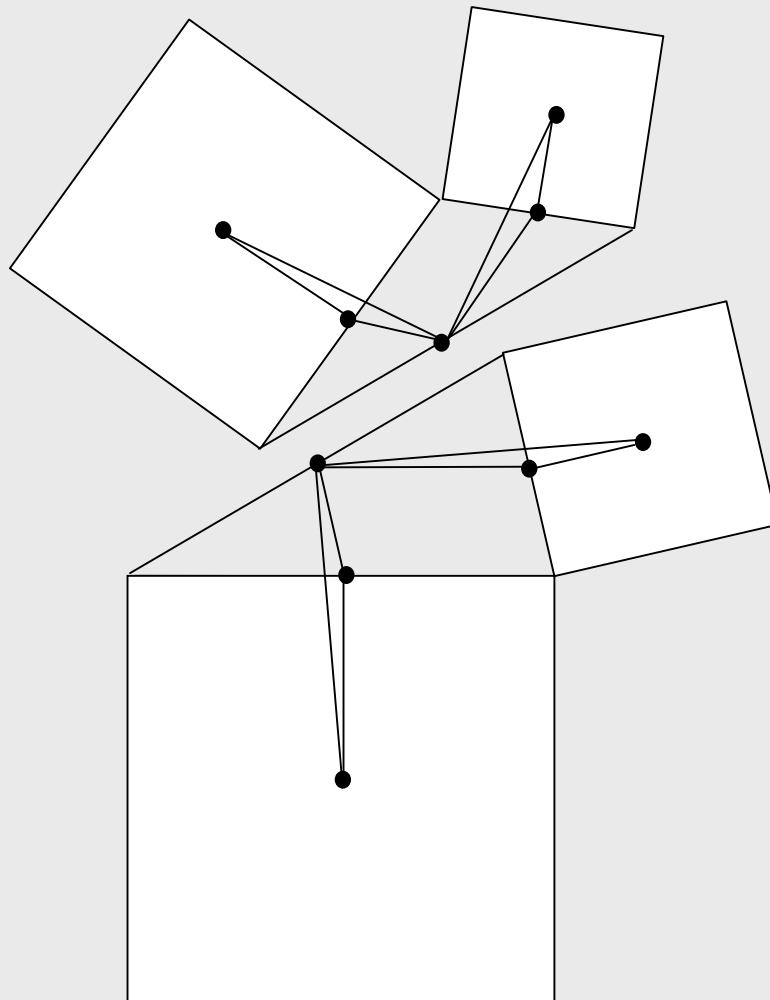
And we have this, too:  $SM = RM$ .

So we can now say that putting together two squares and a triangle the way above and connecting the centers of the squares and the midpoint of the other side of the triangle, we get an **isosceles right triangle**.

Also, we get two identical triangles each of which takes as a side each of the two legs of the right triangle isosceles.

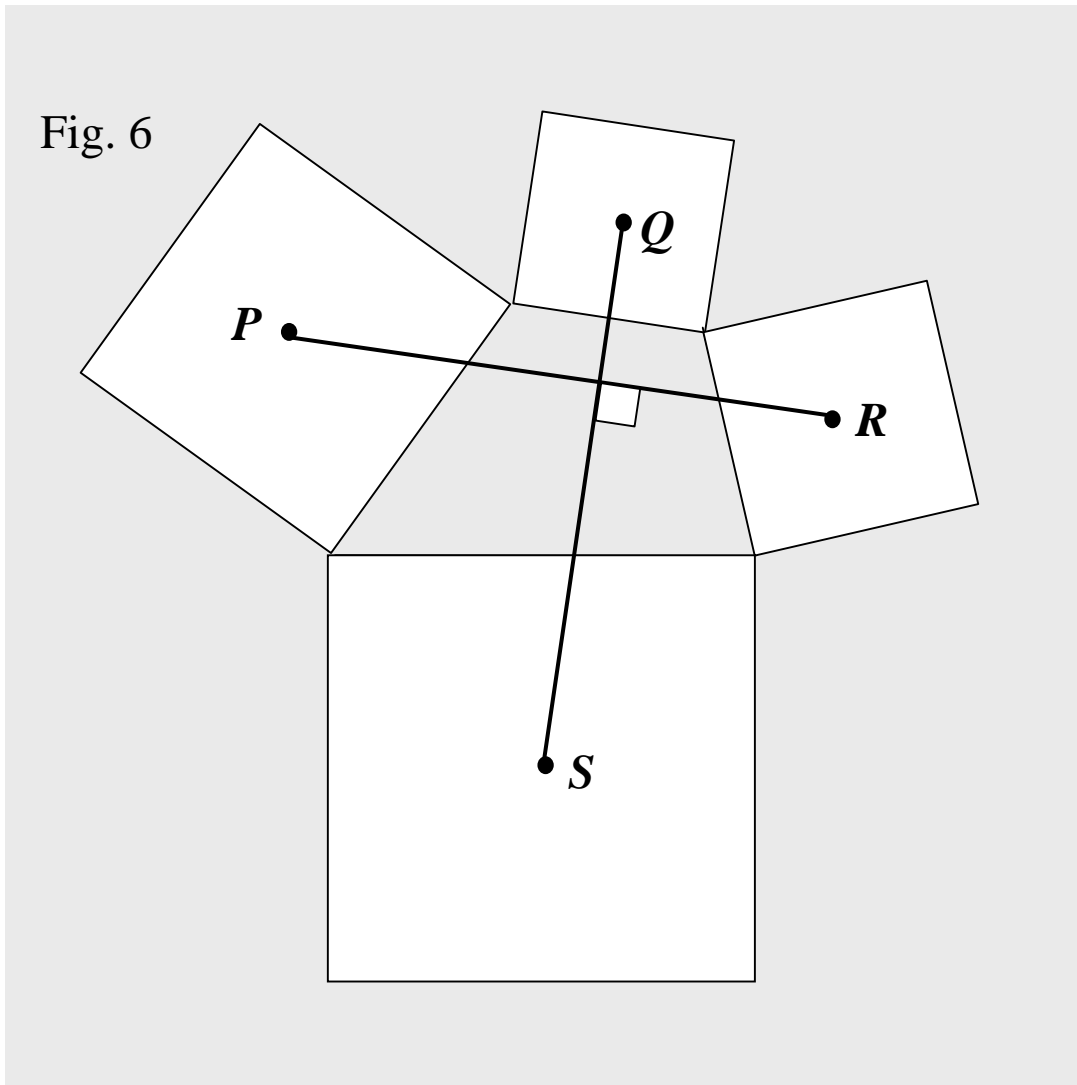
So now, we've got the two sets done, which are as follows.

Fig. 5



And in the next lesson, we will prove the one as follows.

The points  $P$ ,  $Q$ ,  $R$ , and  $S$  are the centers of the squares.



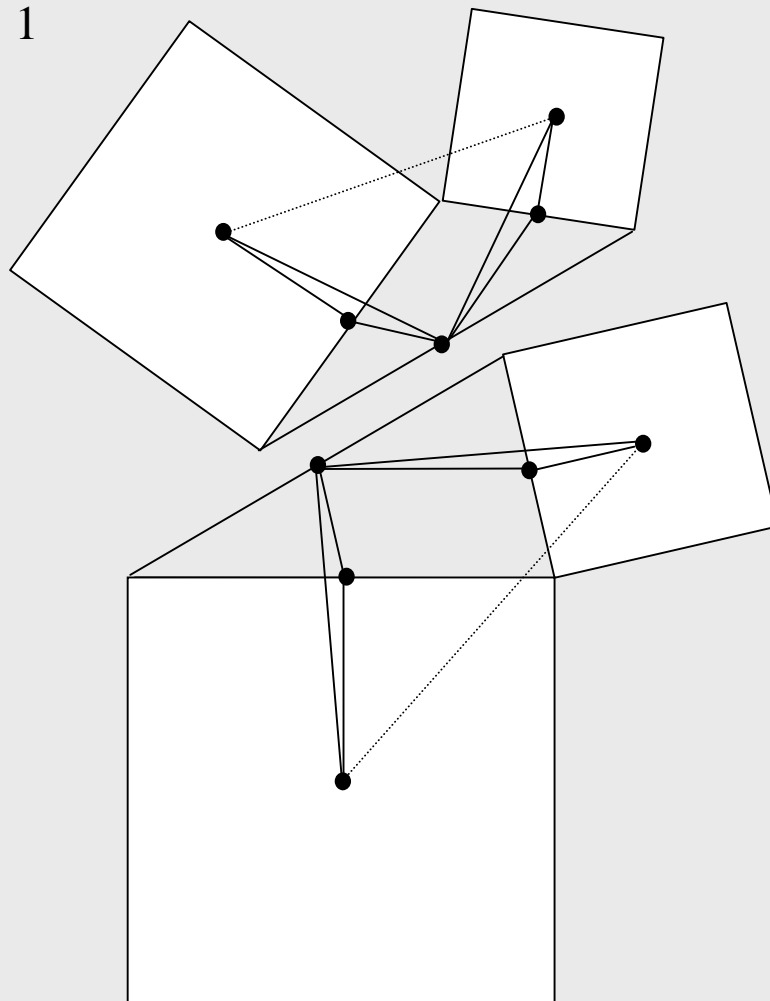
Then, the theorem says that the two line segments  $PR$  and  $QS$  meet at a right angle, that is,  $90^\circ$ , and have the same lengths. And we'll see the proof in the next lesson.

# Van Aubel's Theorem Part 6

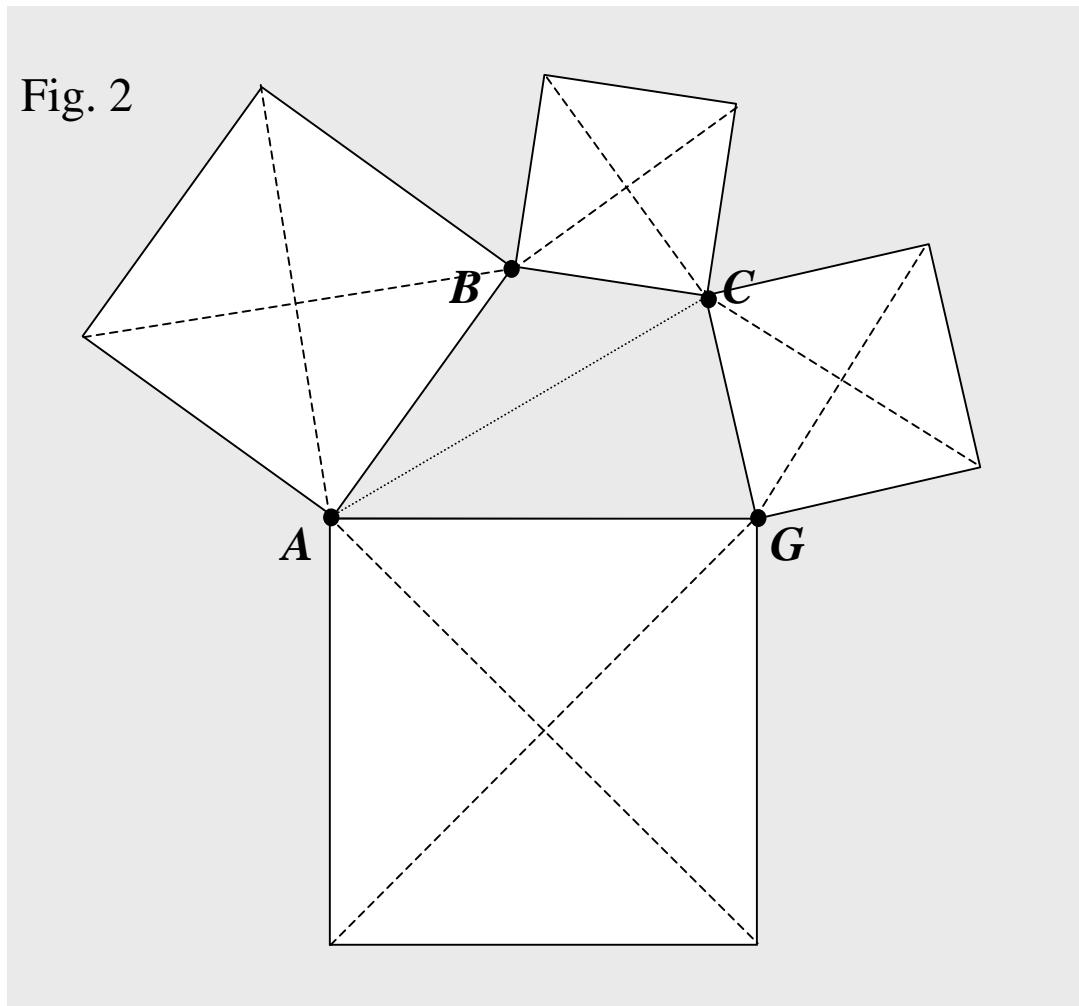
So now, we have shown that in each set of two squares and a triangle, connecting the centers of the squares and the midpoint of the other side of the triangle, we get an ***isosceles right triangle***.

Also, we get two identical triangles each of which takes as a side each of the two legs of the right triangle isosceles.

Fig. 1

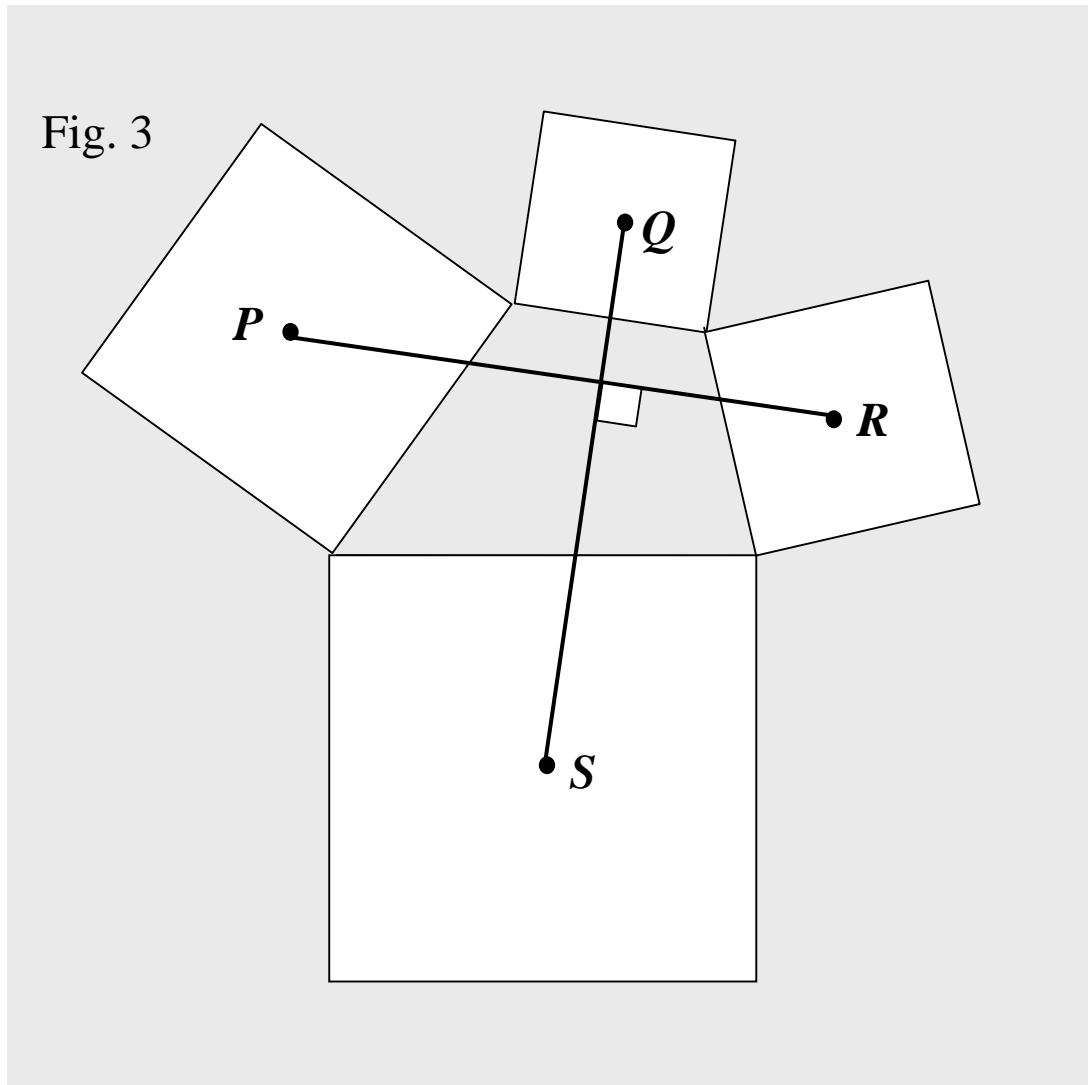


And putting together the two sets, we get the original figure.



Then, as shown above, we get a tetragon  $ABCG$  each side of which is a side of a square. So now, as described in the theorem, connecting with a line segment, each pair of the centers facing each other across the tetragon, we can get a figure as follows.

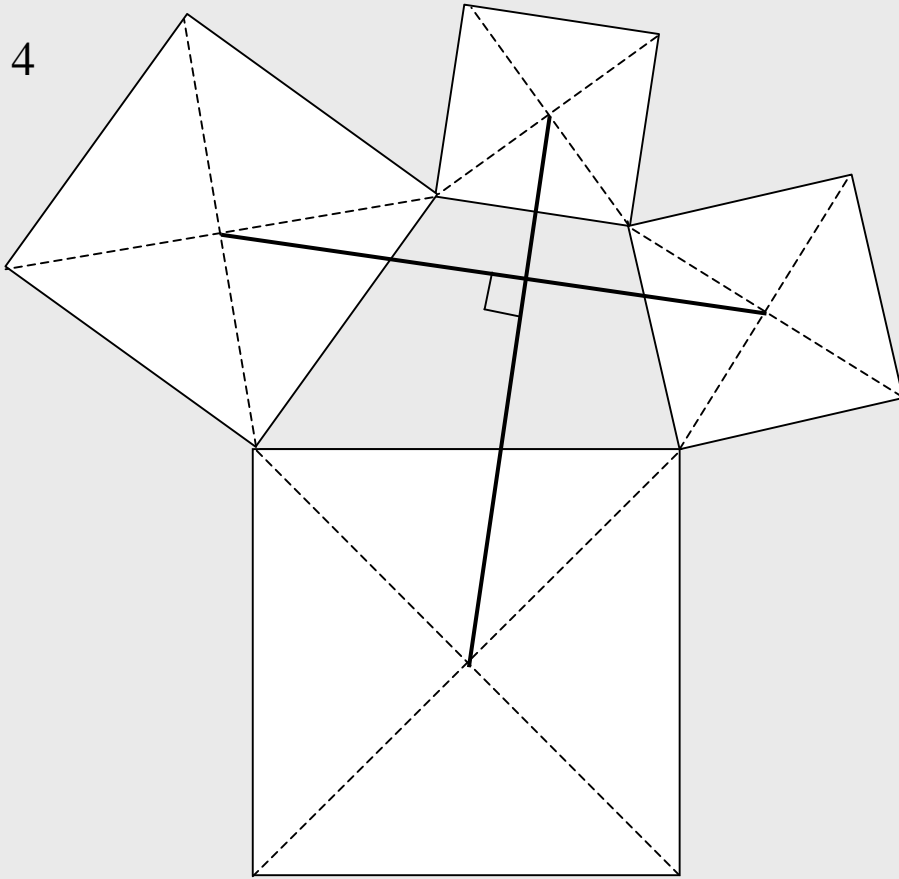
The points  $P$ ,  $Q$ ,  $R$ , and  $S$  are the centers of the squares.



Then, theorem says that the two line segments  $PR$  and  $QS$  meet at a right angle, that is,  $90^\circ$ , and have the same lengths.

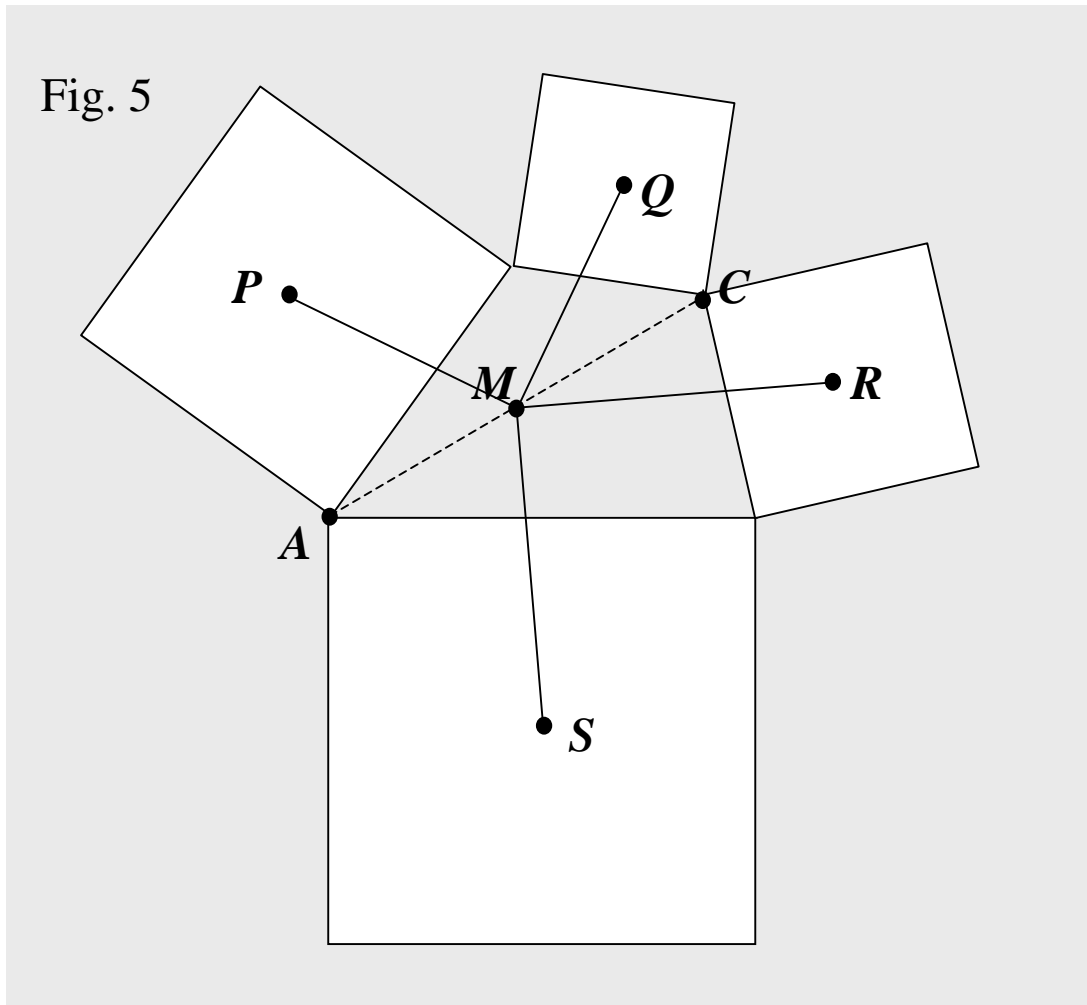
And the same is true of any other tetragon, too. So whatever the tetragon may be, the two line segments are **always** the same as and perpendicular to each other.

Fig. 4



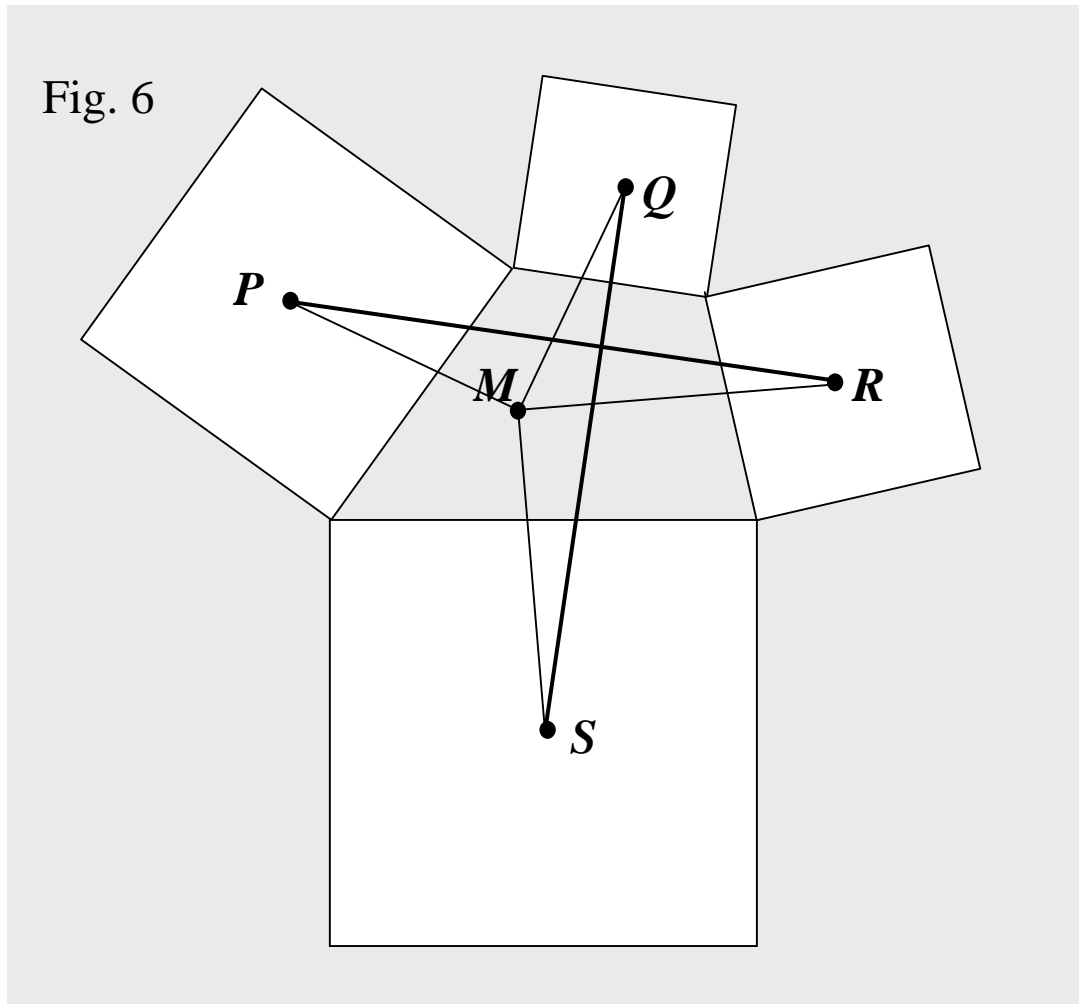
So we are going to see now, **how it is the case**, that is, why always the same and  $90^\circ$  in the story above. And the story is Van Aubel's Theorem.

First, we've shown that in each set of two squares and a triangle, connecting the centers and the midpoint the way below, we get a pair of line segments not only of the same length but perpendicular to each other.



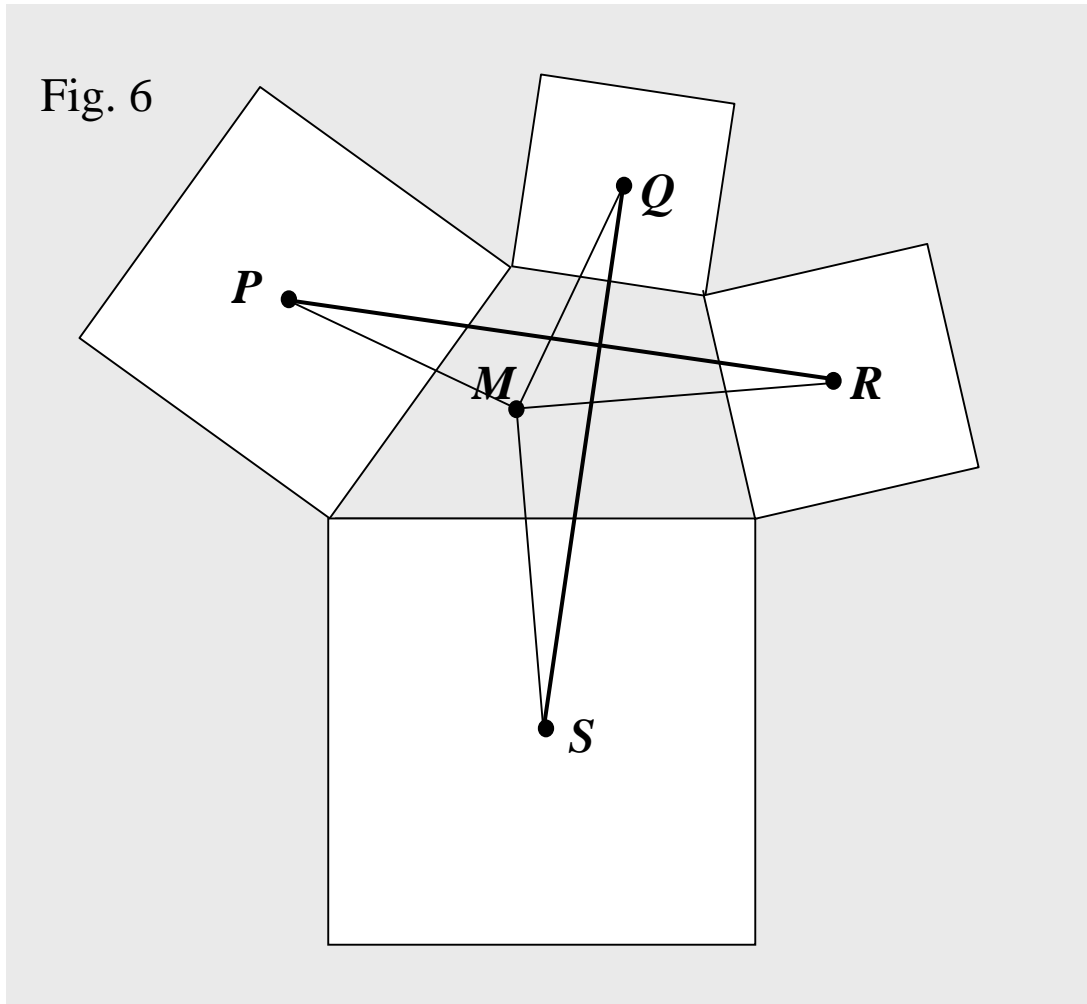
That is, we have these:  $|PM| = |QM|$  and  $|SM| = |RM|$   
 $\angle PMQ = 90^\circ$ , and  $\angle SMR = 90^\circ$ .

So now, as described in the theorem, connecting with a line segment, each pair of the centers facing each other across the tetragon, we can get a figure as follows.



Then, we can see two triangles, one is  $\Delta PMR$ , and the other is  $\Delta QMS$ . And that's not it. Not just two triangles. But we can see something else, also, some interesting things that the two triangles share. What then can the things be?

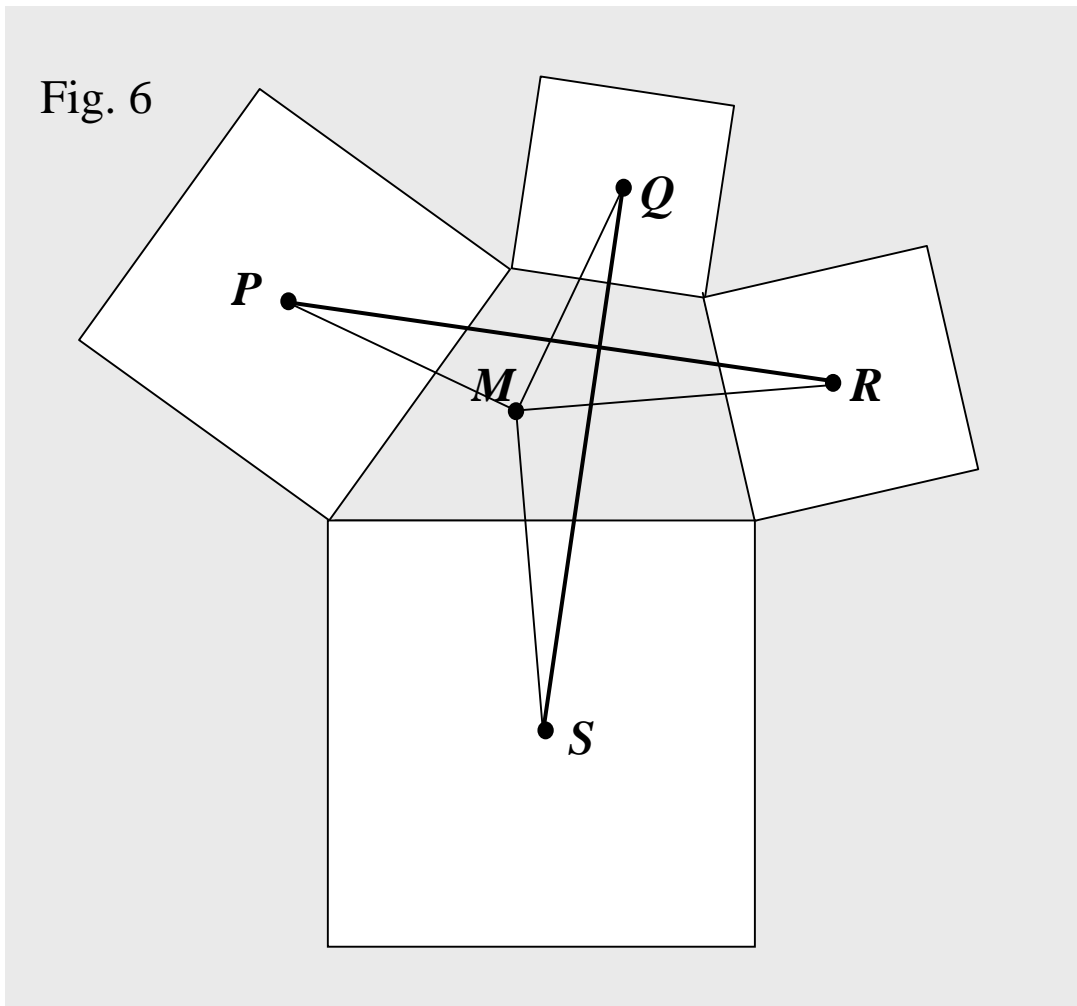
They are two sides. So the two sides are common to both triangles  $\Delta PMR$  and  $\Delta QMS$ . What then are the two?



We know these:  $|PM| = |QM|$  and  $|SM| = |RM|$ .

So the two triangles share these two sides:  $PM$  and  $SM$  or these two:  $QM$  and  $RM$ . What then can we expect?

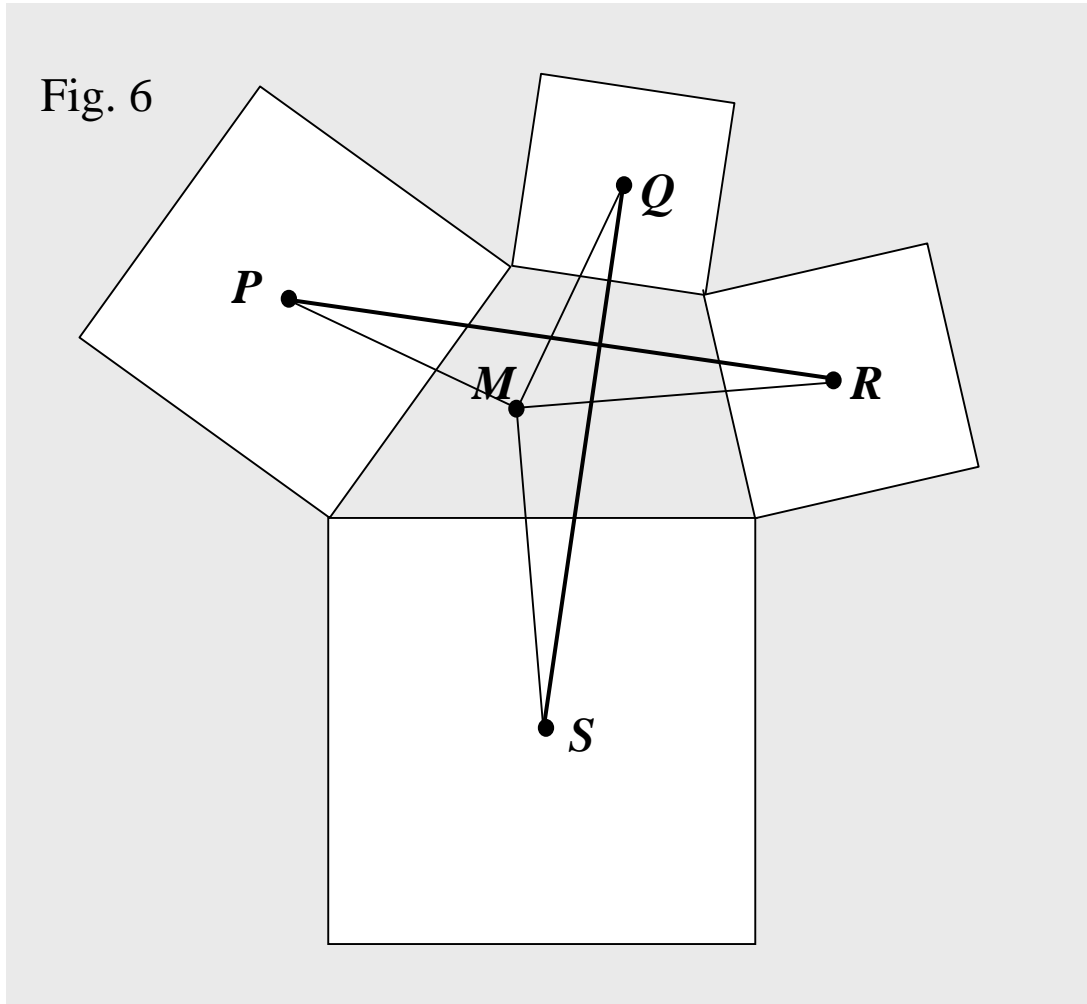
We can expect that the two triangles  $\triangle PMR$  and  $\triangle QMS$  are exactly the same, that is, identical.



So we can expect this:  $\triangle PMR \cong \triangle QMS$ . Proving it thus, we prove that the two line segments  $PR$  and  $QS$  are the same.

How then can we prove it?

As stated earlier, we have a basic that if two triangles share two sides and the angle between the two sides, the two triangles are identical. So what do we need to prove?

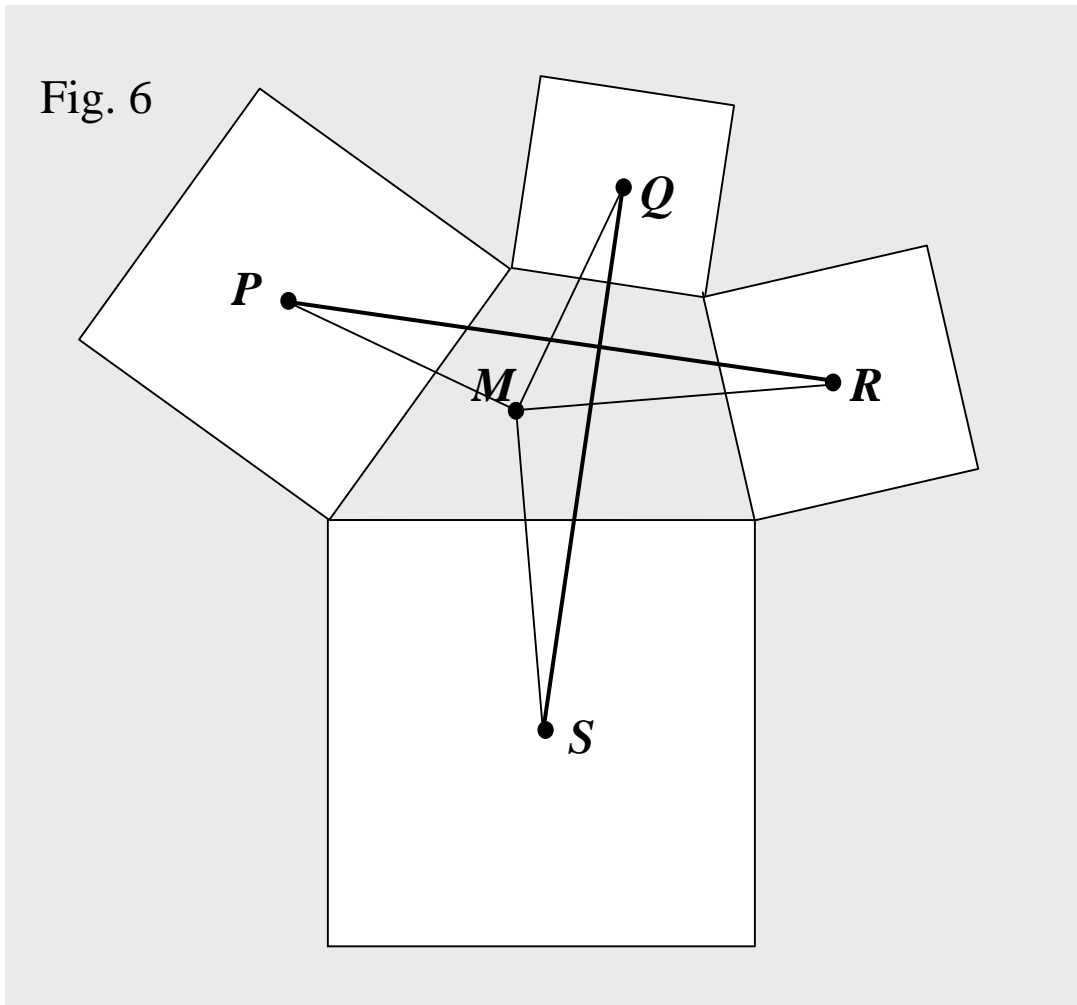


It's an angle. We need to prove that the two triangles share an angle. It's not just an angle, though.

What angle then is it?

The angle between the two sides common to both triangles.

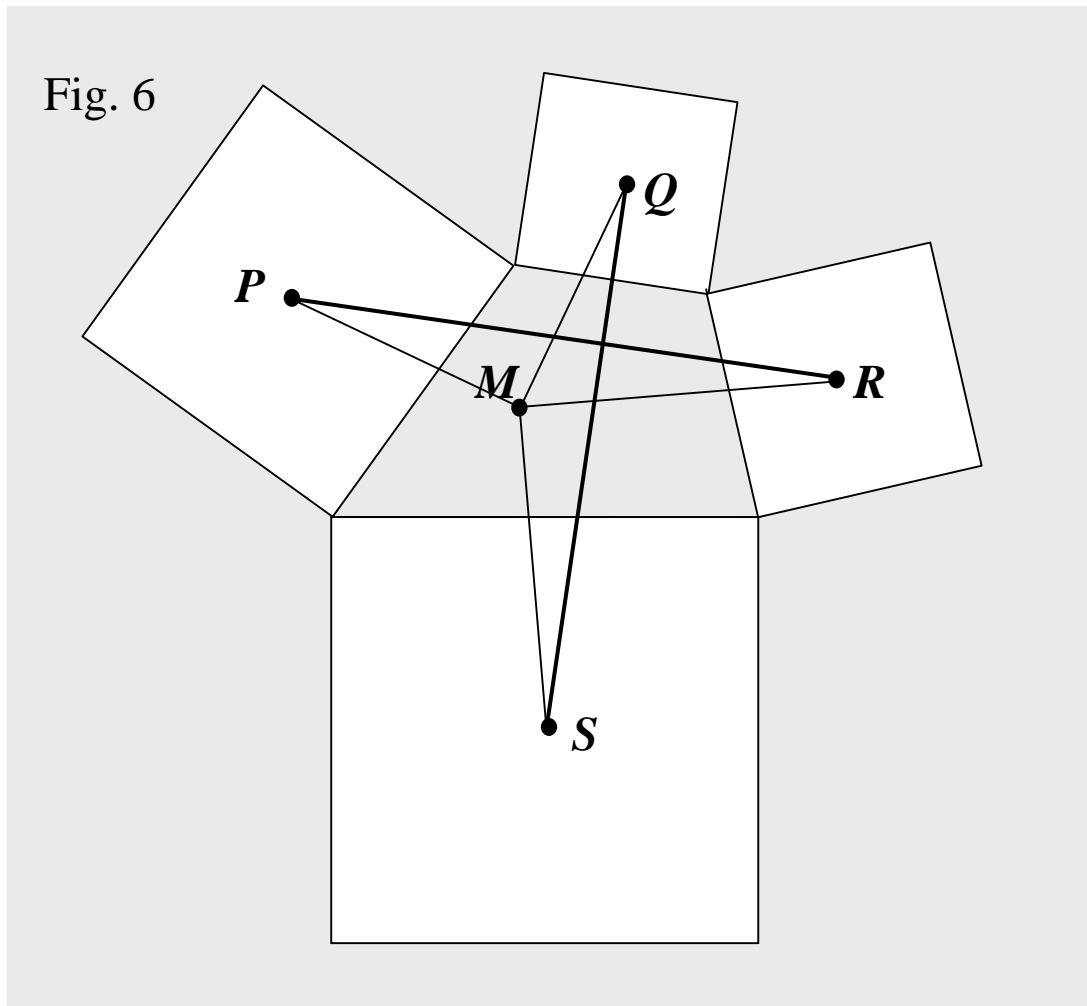
The two can be these:  $PM$  and  $RM$ , or these:  $QM$  and  $SM$ .



The two sides  $PM$  and  $RM$  make  $\angle PMR$ , and the other two sides  $QM$  and  $SM$  make  $\angle QMS$ .

So what can we expect?

We can expect this:  $\angle PMR = \angle QMS$ , which is therefore, we need to prove.



So if proving this:  $\angle PMR = \angle QMS$ , we prove this:

$\triangle PMR \cong \triangle QMS$ , which means, the two line segments  $PR$  and  $QS$  are the same, that is, we get this:  $|PR| = |QS|$ .

And the proof is as follows.

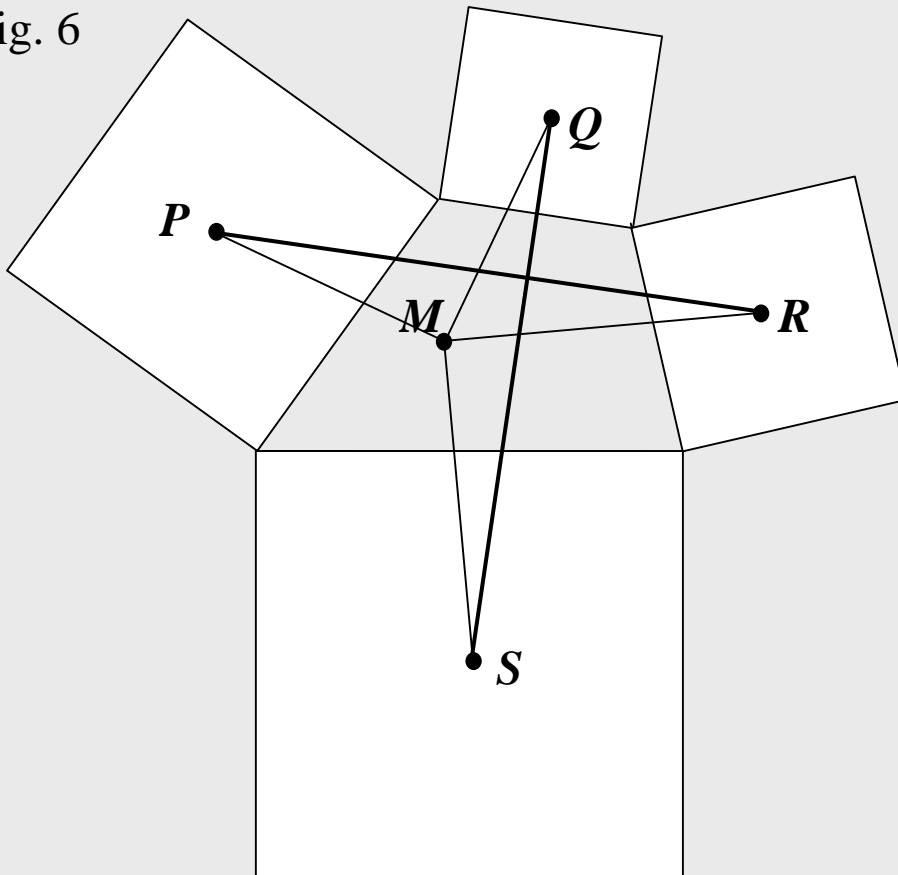
First off, we know these:  $\angle PMQ = 90^\circ$ , and  $\angle SMR = 90^\circ$ .

So we get these:

$$\angle PMR = \angle PMQ + \angle QMR = 90^\circ + \angle QMR.$$

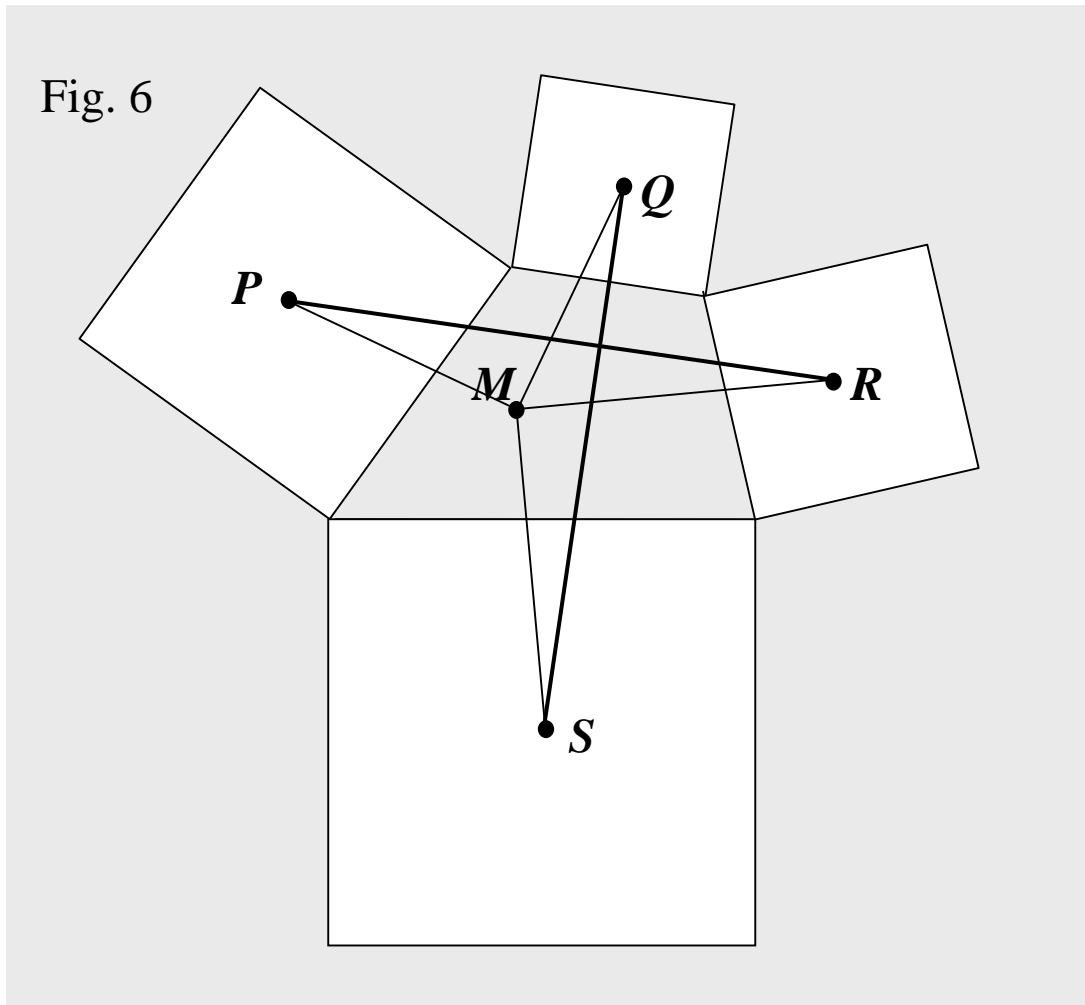
$$\angle QMS = \angle SMR + \angle QMR = 90^\circ + \angle QMR.$$

Fig. 6



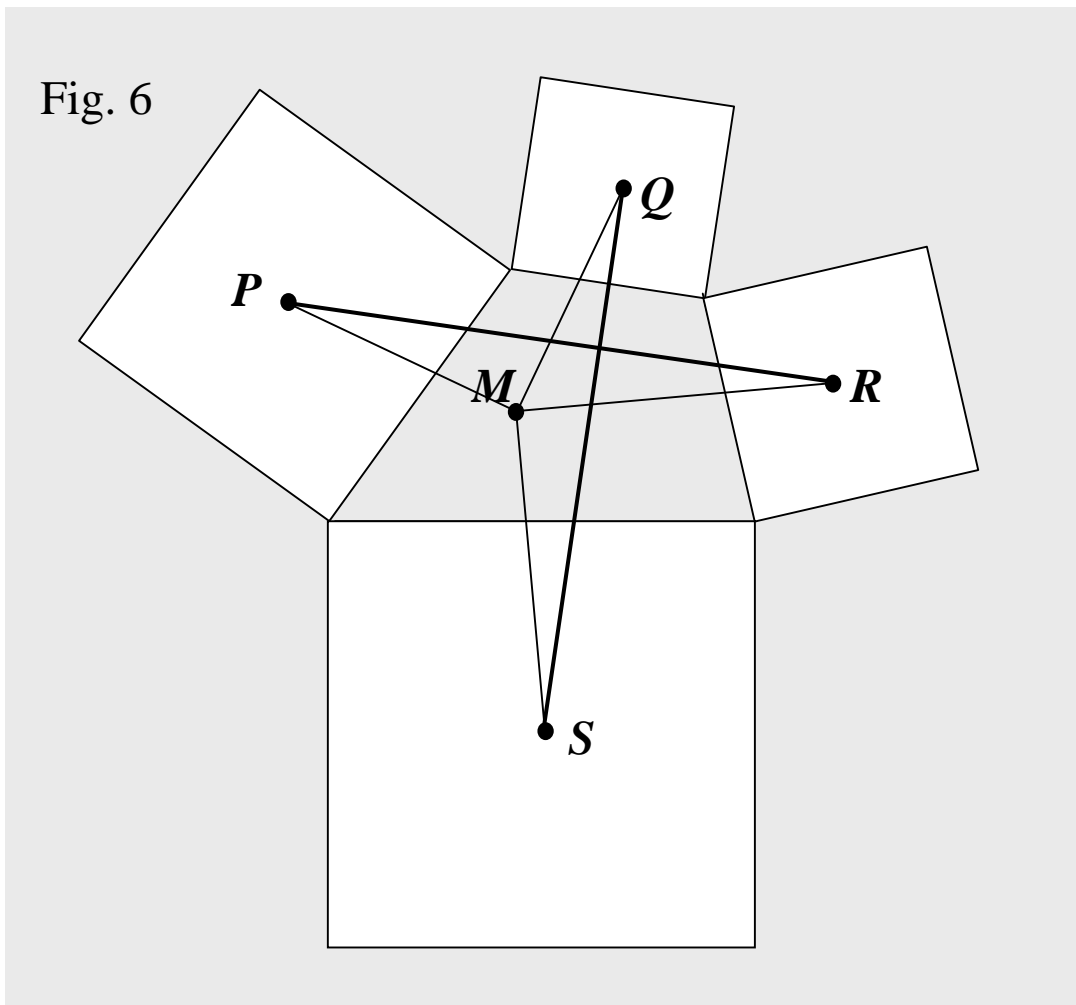
Thus, we get this:  $\angle PMR = \angle QMS$ .

Also, we know these:  $|PM| = |QM|$  and  $|SM| = |RM|$ .



And, as stated earlier, we have the basic that if two triangles share an angle and the two sides making the angle, the two triangles are identical. So we get this:  $\Delta PMR \equiv \Delta QMS$ , which means, the two line segments  $PR$  and  $QS$  are the same, that is, we get this:  $|PR| = |QS|$ .

What then about the angle between  $PR$  and  $QS$ ?

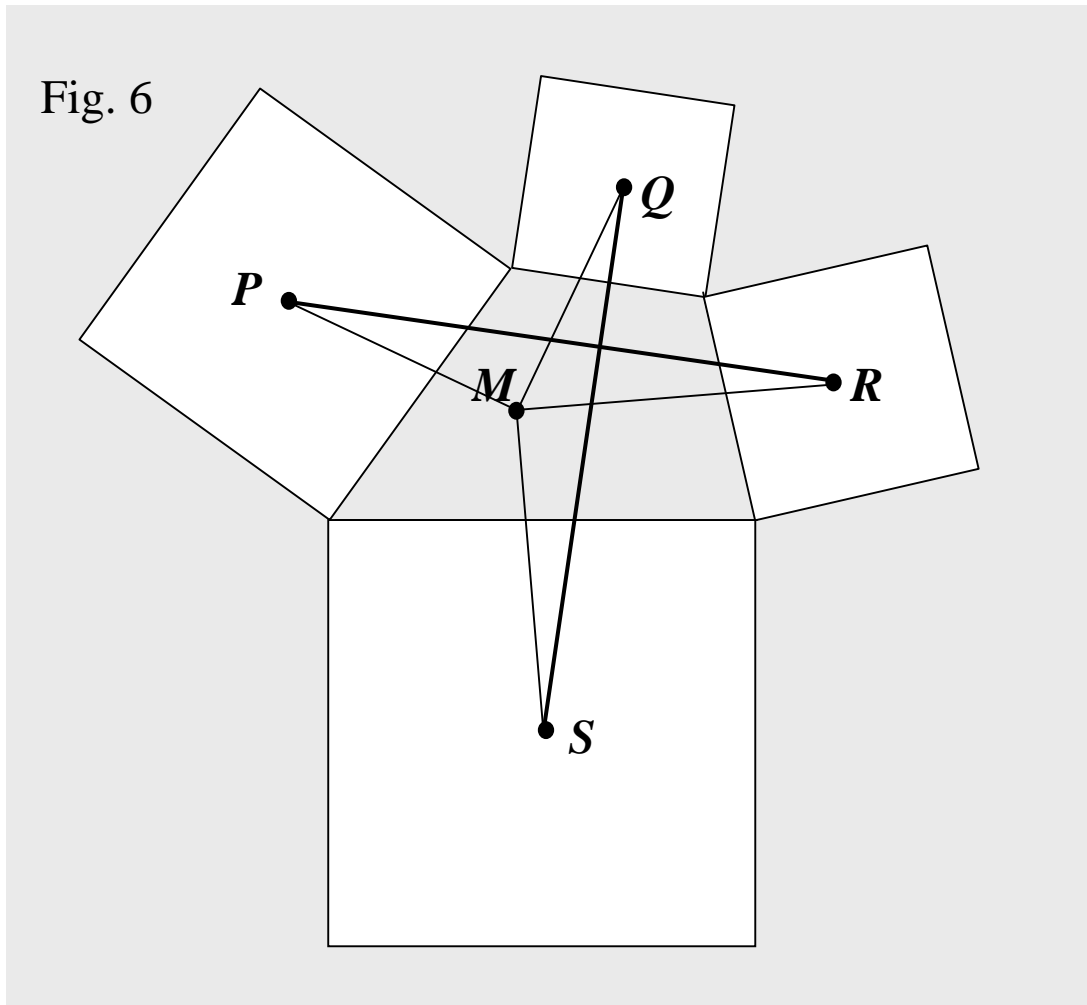


We know that  $\Delta PMR$  is identical to  $\Delta QMS$ .

Also, we know these:  $\angle PMR = \angle QMS$ ,

$|PM| = |QM|$ ,  $|SM| = |RM|$ ,  $|PR| = |QS|$

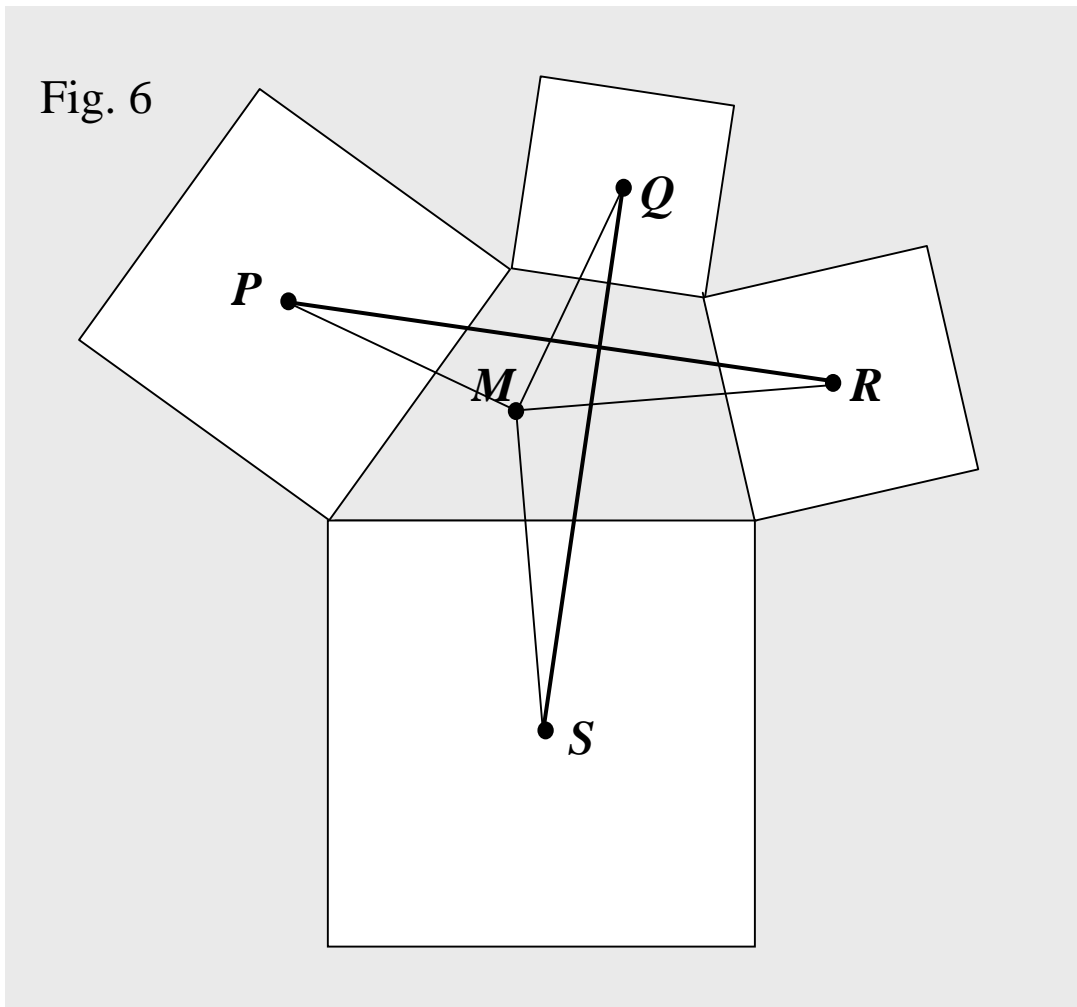
And we have these:  $\angle PMQ = 90^\circ$ , and  $\angle SMR = 90^\circ$ .



Turning thus,  $\Delta PMR$  clockwise  $90^\circ$  about the point  $M$ , we can make  $\Delta PMR$  cover exactly  $\Delta QMS$ .

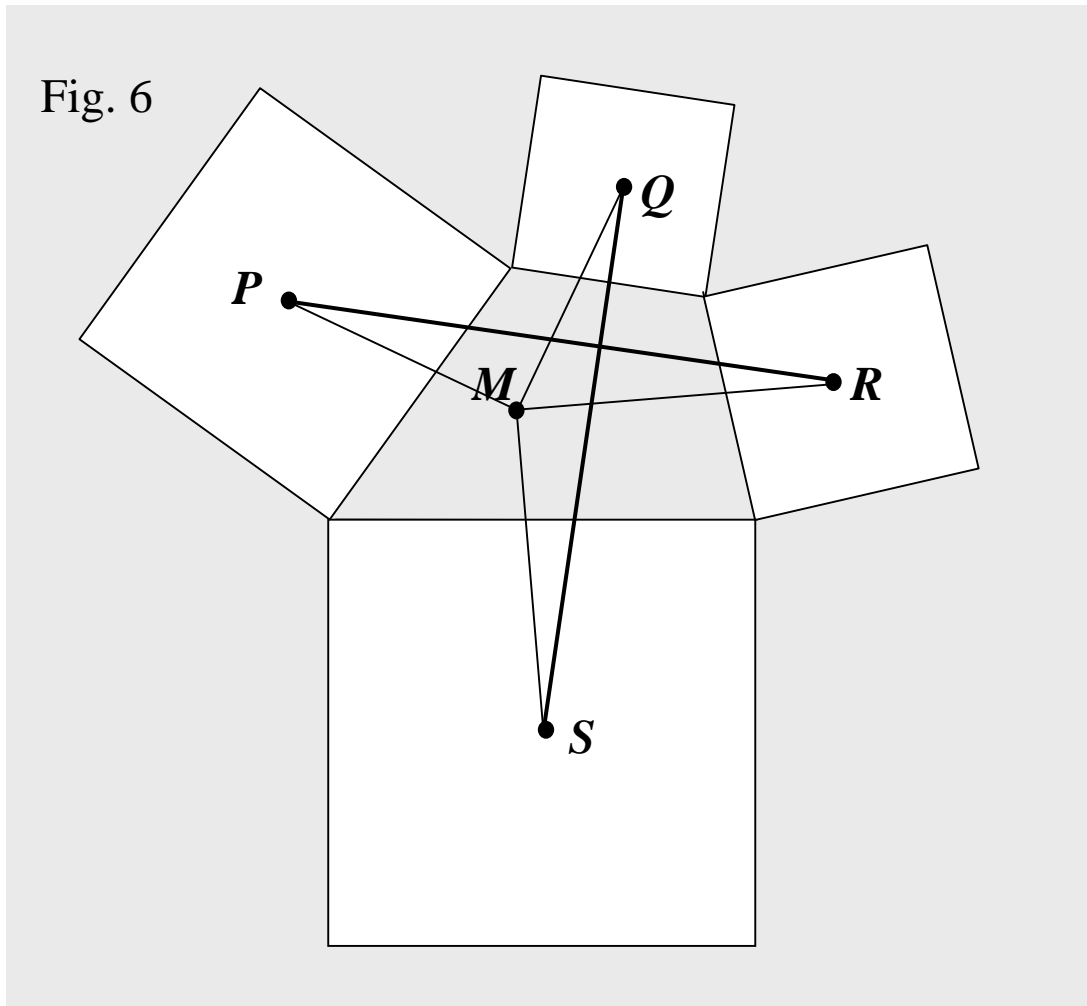
That is to say that turning  $\Delta PMR$  clockwise  $90^\circ$  about the point  $M$ , we get  $\Delta QMS$ .

And if turning a triangle, we turn every side of the triangle the same amount in angle and the same manner.



So turning  $\triangle PMR$  clockwise  $90^\circ$  about the point  $M$ , we get to turn all the sides  $PM$ ,  $RM$ , and  $PR$  clockwise  $90^\circ$  about the point  $M$  at the same time.

Then, the side  $PM$  will be the side  $QM$ , the side  $PR$  will be the side  $QS$ , and the side  $RM$  will be the side  $SM$ .



And we have these:  $\angle PMQ = \angle SMR = 90^\circ$ .

That is, we have these:  $PM \perp QM$ , and  $SM \perp RM$ , so we get this, too:  $PR \perp QS$ .

Thus, we can now say that  $|PR| = |QS|$ , and  $PR \perp QS$ .